

# On Ordering Spatio-Temporal Sequences to meet Transition Constraints

Michael Sioutis and Jean-François Condotta and Yakoub Salhi and Bertrand Mazure

Université d’Artois, CRIL CNRS UMR 8188, Lens, France

{sioutis, condotta, salhi, mazure}@cril.fr

David A. Randell

School of Dentistry, College of Medical and Dental Sciences, University of Birmingham, UK

d.a.randell@bham.ac.uk

## Abstract

*Time* and *space* are fundamental concepts of study in Artificial Intelligence and, in particular, Knowledge Representation. In this paper, we investigate the task of ordering a temporal sequence of qualitative spatial configurations to meet certain transition constraints. This ordering is constrained by the use of conceptual neighbourhood graphs defined on qualitative spatial constraint languages. In particular, we show that the problem of ordering a sequence of qualitative spatial configurations to meet such transition constraints is  $\mathcal{NP}$ -complete for the well known languages of RCC-8, Interval Algebra, and Rectangle Algebra. Our results lie within the area of Graph Traversal and allow for many practical and diverse applications, such as identifying optimal routes in mobile robot navigation, modelling changes of topology in biological processes, and computing sequences of segmentation steps used in image processing algorithms.

## 1 Introduction

*Time* and *space* are fundamental cognitive concepts that have been the focus of study in many scientific disciplines, including Artificial Intelligence and, in particular, Knowledge Representation. In this context, an emphasis has been made on qualitative spatiotemporal reasoning, which abstracts from numerical quantities of space and time using qualitative values instead (e.g., earlier, bigger, left of). The conciseness of the representational language used in the qualitative approach provides a promising framework that further boosts research and applications in spatiotemporal reasoning [Hazrika, 2012; Wolter and Zakharyashev, 2003].

In this paper, we focus on a particular spatiotemporal reasoning problem that lies within the area of Graph Traversal, which is one of the oldest areas of inquiry in Graph Theory. Graph Traversal commonly deals with visiting all the nodes in a graph in a particular manner, updating and/or checking their values along the way. We are interested in a problem related to the Hamiltonian path problem for a given graph, which is the graph traversal problem of finding a path in the graph that visits each vertex exactly once. Hamiltonian path related problems naturally extend into use cases where routes

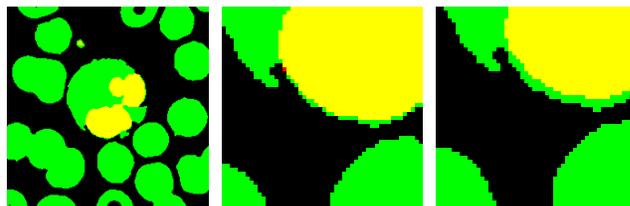


Figure 1: Left: segmented cell bodies (green), lobulated cell nuclei (yellow and red) and background (black), Middle: segmented cell nucleus extending outside border of host cell (red pixels), Right: the result of applying a morphological erosion operator; in this case the original *partially overlaps* relation changes to *proper part*

need to be ordered or optimised, minimising the traversal of paths and vertices already visited. This abstraction has many practical and diverse applications, from identifying optimal routes in mobile robot navigation, to modelling changes of topology in biological processes and computing sequences of segmentation steps used in image processing algorithms. All these application examples can be modelled as a sequence of successive states where we look for ways to order the states so that an assumed set of a priori constraints are satisfied. For example, in the case of a phagocyte ingesting food, one constraint may be that the food has to be part of a food vacuole in the animal before it can be digested and absorbed.

After introducing the general context of the problem we wish to study here, we specify our *problem of interest* as a Hamiltonian path related problem where we want to order a sequence of qualitative spatial configurations to meet certain transition constraints. This ordering is constrained by the use of conceptual neighbourhood graphs defined on qualitative spatial constraint languages. For this problem, we consider several well known qualitative constraint languages, such as RCC-8 [Randell *et al.*, 1992], Interval Algebra (IA) [Allen, 1983], and Rectangle Algebra (RA) [Guesgen, 1989]. In particular, RCC-8 encodes topological relations between two regions that are non-empty regular subsets of some topological space, IA encodes relative position relations between intervals, and RA encodes relative position and containment relations between multi-dimensional objects.

We have already claimed that this abstraction has practical applications and now give a detailed example to better

motivate the subject of our paper. In [Randell *et al.*, 2013] the authors use a discrete version of the spatial logic RCC (from which the constraint language RCC-8 is derived) called DM (for Discrete Mereotopology) to model the topological organization of segmented cells and their parts and cellular structure in tissue. The domain model assumes an a priori constraint that cell nuclei form parts of their host cells, however in the example shown in Figure 1 the RCC-8 relation returned is *partially overlaps* and not *proper part*. There are several reasons why this scenario may happen in practice, e.g., if the regions initially segmented out as cell nuclei are being over-segmented, or variations in the histological stain density results in a less than optimal threshold level being selected. The result means the labelled regions extracted from the image cannot be a model. The task then is to repair the segmentation to restore consistency and/or optimise the sequence of segmentation steps needed. As such, a conceptual neighbourhood graph for DM is used to encode legal topological transitions, and successive states from a start to end state are generated, and then optimised. Paths through the network are then cashed out as a series of image processing segmentation steps. A single histological image may have many hundreds of cells, and the generation of symbolic models may or may not be realised in an actual image. Moreover, some segmentation operations on regions will reduce their size and may fragment a region into sub-parts, or separated regions that increase their size may merge, so the computational task of finding an optimal segmentation model can easily grow in complexity.

Two closely related contributions that deal with sequences of qualitative spatial or temporal configurations consist of the works of Westphal *et al.* in [Westphal *et al.*, 2013] and Cui *et al.* in [Cui *et al.*, 1992]. In both of these papers, qualitative configurations extracted follow a predefined ordering; where all pairs of consecutive qualitative configurations in the sequence produced, meet certain transition constraints with respect to an assumed conceptual neighbourhood graph. In our case, our knowledge base already comprises a set of qualitative configurations, and the problem is that of finding an ordering of those qualitative configurations when positioned in a sequence, such that all the pairs of consecutive qualitative configurations in the ordered sequence meet the aforementioned transition constraints. Thus, we define a novel problem in the context of qualitative spatiotemporal reasoning, whose computational properties we are the first to study. In particular, we make the following contribution: we consider a sequence of qualitative spatial configurations of RCC-8, IA, or RA, and show that it is  $\mathcal{NP}$ -complete to order the configurations in a way such that the transition constraints are met with respect to the conceptual neighbourhood graph of the considered language.

The paper is organized as follows. In Section 2 we introduce the notions of a qualitative constraint language, a qualitative constraint network, and a conceptual neighbourhood graph. Section 3 is our main section where we introduce the notions of a qualitative spatiotemporal sequence and a transition graph which encodes certain transition constraints, but also define our main problem, that of obtaining a desired ordering of the configurations in a given sequence, and provide

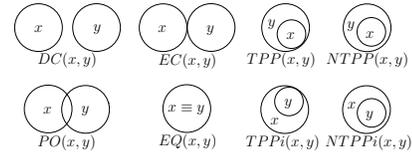


Figure 2: 2D examples of the RCC-8 base relations

complexity results for different variations of it. Finally, in Section 4 we conclude and discuss future work.

## 2 Qualitative constraint networks and conceptual neighbourhood graphs

A (binary) qualitative temporal or spatial constraint language [Renz and Ligozat, 2005] is based on a finite set  $B$  of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain  $D$ , called the set of base relations. The base relations of set  $B$  of a particular qualitative constraint language can be used to represent the definite knowledge between any two entities with respect to the given level of granularity.  $B$  contains the identity relation  $\text{Id}$ , and is closed under the converse operation ( $^{-1}$ ). Indefinite knowledge can be specified by unions of possible base relations, and is represented by the set containing them. Hence,  $2^B$  represents the total set of relations.  $2^B$  is equipped with the usual set-theoretic operations union and intersection, the converse operation, and the weak composition operation denoted by symbol  $\diamond$  [Renz and Ligozat, 2005]. We note that the notion of a qualitative constraint language is not consistent in literature, an issue which is nicely explained in the work of Dylla *et al.* in [Dylla *et al.*, 2013]. In this paper we consider a qualitative constraint language to be a *relation algebra* [Dylla *et al.*, 2013], which is the case for the most well known calculi such as RCC-8 [Randell *et al.*, 1992], Interval Algebra (IA) [Allen, 1983], and Rectangle Algebra (RA) [Guesgen, 1989] (where its base relations can be embedded by an isomorphism into the base relations of IA).

**Example.** As an example, the qualitative spatial constraint language RCC-8 [Randell *et al.*, 1992] comprises the set of base relations  $\{DC$  (disconnected),  $EC$  (externally connected),  $PO$  (partially overlaps),  $TPP$  (tangential proper part),  $NTPP$  (non-tangential proper part),  $TPPi$  (tangential proper part inverse),  $NTPPi$  (non-tangential proper part inverse),  $EQ$  (equals) $\}$ , with  $EQ$  being the identity relation, as depicted in Figure 2.

Likewise, there exist other qualitative spatial constraint languages, such as the Interval Algebra (IA) [Allen, 1983] and the Rectangle Algebra (RA) [Guesgen, 1989], with their own sets of base relations.

In what follows, if  $\mathcal{L}$  is a qualitative constraint language, we will denote by  $B_{\mathcal{L}}$  its set of base relations. Qualitative temporal or spatial constraint configurations can be formulated as qualitative constraint networks (QCNs) as follows:

**Definition 1** *Given a qualitative constraint language  $\mathcal{L}$ , a QCN  $\mathcal{N}$  of  $\mathcal{L}$  is a pair  $(V, C)$  where:  $V$  is a non-empty finite set of variables;  $C$  is a mapping that associates a relation  $C(v, v') \in 2^{B_{\mathcal{L}}}$  to each pair  $(v, v')$  of  $V \times V$ .  $C$  is such that  $C(v, v) = \{\text{Id}\}$  and  $C(v, v') = (C(v', v))^{-1}$ .*

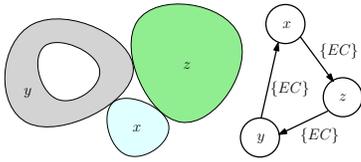


Figure 3: A RCC-8 configuration (left) along with its QCN describing it (right)

An example of a QCN of RCC-8 is depicted in Figure 3. Note that we always regard a QCN as a complete network. In what follows, given a QCN  $\mathcal{N} = (V, C)$  and  $v, v' \in V$ ,  $\mathcal{N}[v, v']$  will denote the relation  $C(v, v')$ . Given a QCN  $\mathcal{N} = (V, C)$  defined in some qualitative constraint language  $\mathcal{L}$ , we have the following definitions:  $\mathcal{N}$  is said to be *trivially inconsistent* iff  $\exists v, v' \in V$  with  $\mathcal{N}[v, v'] = \emptyset$ . A *solution* of  $\mathcal{N}$  is a mapping  $\sigma$  defined from  $V$  to the domain  $D$ , yielding a spatial configuration, such that for every pair  $(v, v')$  of variables in  $V$ ,  $(\sigma(v), \sigma(v'))$  can be described by  $\mathcal{N}[v, v']$ , i.e., there exists a base relation  $b \in \mathcal{N}[v, v']$  such that the base relation defined by  $(\sigma(v), \sigma(v'))$  is  $b$ .

**Definition 2** A QCN  $\mathcal{N}$  is satisfiable iff it admits a solution.

A sub-QCN  $\mathcal{N}'$  of  $\mathcal{N}$ , is a QCN  $(V, C')$  such that  $\mathcal{N}'[v, v'] \subseteq \mathcal{N}[v, v'] \forall v, v' \in V$ . If  $b$  is a base relation, then  $\{b\}$  is a singleton relation. An *atomic* QCN is a QCN where each constraint is a singleton relation. Given a solution  $\sigma$  of  $\mathcal{N}$ , a *scenario*  $\mathcal{N}(\sigma)$  of  $\mathcal{N}$  is an atomic satisfiable sub-QCN of  $\mathcal{N}$ , such that  $\forall v, v' \in V$ ,  $\mathcal{N}(\sigma)[v, v']$  is defined by the base relation defined by  $(\sigma(v), \sigma(v'))$ . A subclass of relations is a set  $\mathcal{A} \subseteq 2^{\mathcal{B}_{\mathcal{L}}}$  closed under converse, intersection, and weak composition. In what follows, all the considered subclasses will contain the singleton relations and the universal relation  $\mathcal{B}_{\mathcal{L}}$  of  $2^{\mathcal{B}_{\mathcal{L}}}$ .

**Definition 3** Given a qualitative constraint language  $\mathcal{L}$ , a subclass  $\mathcal{A} \subseteq 2^{\mathcal{B}_{\mathcal{L}}}$  is a tractable subclass if the satisfiability problem for a QCN  $\mathcal{N}$  of  $\mathcal{L}$  comprising only relations from  $\mathcal{A}$  is tractable. A subclass  $\mathcal{A} \subseteq 2^{\mathcal{B}_{\mathcal{L}}}$  is a maximal tractable subclass if there is no other tractable subclass properly containing  $\mathcal{A}$ .

A QCN  $\mathcal{N}$  is  $\diamond$ -consistent or closed under weak composition iff  $\forall v, v', v'' \in V$  we have that  $\mathcal{N}[v, v'] \subseteq \mathcal{N}[v, v''] \diamond \mathcal{N}[v'', v']$ . Given a QCN  $\mathcal{N} = (V, C)$ ,  $\diamond$ -consistency can be applied in  $O(|V|^3)$  time [Renz and Ligozat, 2005]. The constraint graph of a QCN  $\mathcal{N} = (V, C)$  is the graph  $(V, E)$ , denoted by  $G(\mathcal{N})$ , for which we have that  $(v, v') \in E$  iff  $\mathcal{N}[v, v'] \neq \mathcal{B}$ .

Checking the satisfiability of a QCN is  $\mathcal{NP}$ -complete in general for the most well known calculi such as RCC-8, IA, and RA. However, there exist maximal tractable subclasses for those calculi for which the satisfiability problem for not trivially inconsistent and  $\diamond$ -consistent QCNs comprising relations only from one of those subclasses becomes tractable, as noted earlier. For example, the maximal tractable subclasses for RCC-8 are the classes  $\mathcal{H}_8$ ,  $\mathcal{C}_8$ , and  $\mathcal{Q}_8$  [Renz and Nebel, 2001]. For other calculi, their whole class of relations is tractable, as is the case with Point Algebra (PA) [van Beek, 1992], i.e.,  $2^{\mathcal{B}_{\text{PA}}}$  is the maximal tractable subclass of

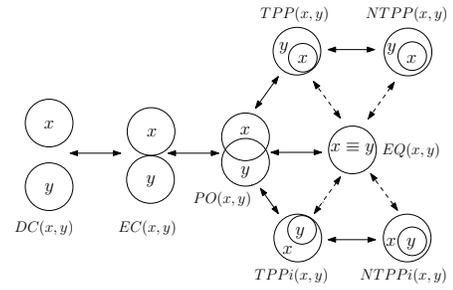


Figure 4: A conceptual neighbourhood graph of RCC-8

PA. Regarding PA in particular, checking the satisfiability of a QCN of PA can be done in  $O(|V|^2)$  time with a dedicated algorithm presented in [van Beek, 1992, chap. 3]. However,  $\diamond$ -consistency, as a more general approach, is still suitable for deciding the satisfiability of a QCN of PA.

The notion of conceptually neighbouring relations in some qualitative constraint language is strongly related to the continuity and proximity that these relations might exhibit. In particular, we recall the following definition from [Freksa, 1991]:

**Definition 4 ([Freksa, 1991])** Given a qualitative constraint language  $\mathcal{L}$ , we have that two base relations  $b(u, v)$  and  $b'(u, v)$ , with  $b, b' \in \mathcal{B}_{\mathcal{L}}$  and  $u, v$  being two entities, are conceptual neighbours if they are proximal and can be directly transformed into one another by continuous deformation (e.g., in shape, size, or position) of entities  $u$  and  $v$ .

**Example.** As an example, in RCC-8 the base relations  $DC(x, y)$  and  $EC(x, y)$  are conceptual neighbours since a continuous movement of the spatial entity  $x$  towards spatial entity  $y$  may cause a direct transition from relation  $DC(x, y)$  to relation  $EC(x, y)$ . The relations  $DC(x, y)$  and  $PO(x, y)$  are not conceptual neighbours since a transition between those relations must go through relation  $EC(x, y)$ .

Clearly, by Definition 4 it follows that every base relation is a conceptual neighbour of itself, however, we do not depict any loops in our graphs to follow for simplicity.

Conceptually neighbouring relations in any given qualitative constraint language  $\mathcal{L}$  can be captured with a conceptual neighbourhood graph, which is defined as follows:

**Definition 5 ([Freksa, 1991])** Given a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph of  $\mathcal{L}$  is a graph  $\Gamma = (\mathcal{B}_{\mathcal{L}}, E)$  where  $E = \{(b(u, v), b'(u, v)) \mid b, b' \in \mathcal{B}_{\mathcal{L}} \text{ and } u, v \text{ being two entities; and } b(u, v) \text{ and } b'(u, v) \text{ being conceptual neighbours}\}$ .

Conceptual neighbourhood graphs can be established for all qualitative constraint languages, a subset of which can be found in [Freksa, 1991; Santos and Moreira, 2009; Egenhofer, 2010]. It is important to note that conceptual neighbourhood graphs are not unique for every qualitative constraint language as they can be subject to further restrictions, such as constraints subject to user preference, or restrictions on deformation.

**Example.** As an example, the conceptual neighbourhood graph of RCC-8 is depicted in Figure 4. The dashed edges

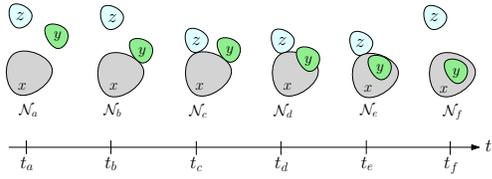


Figure 5: Example of a spatiotemporal sequence based on RCC-8

represent transitions of base relations that are not allowed if we require that regions do not change size.

### 3 Qualitative spatiotemporal sequences and transition graphs

In general, a spatial QCN, as described in Section 2, constitutes a static spatial configuration in some domain, over a set of spatial entities. To be able to describe a spatial configuration that changes over time, we can define the notion of a qualitative spatiotemporal sequence, which is nothing more than a sequence of spatial QCNs. The ordering of the QCNs in the aforementioned sequence constitutes a timeline that allows us to view how a spatial configuration evolves over time. We can define a qualitative spatiotemporal sequence (QSS) as follows:

**Definition 6** Given a qualitative constraint language  $\mathcal{L}$ , a QSS  $\mathcal{S}$  of  $\mathcal{L}$  is a sequence  $(\mathcal{N}_1 = (V, C_1), \mathcal{N}_2 = (V, C_2), \dots, \mathcal{N}_k = (V, C_k))$  of  $k$  QCNs of  $\mathcal{L}$  over a set of  $n$  variables  $V$ , for some integers  $k$  and  $n$ .

An atomic QSS is a QSS that comprises only atomic QCNs. Further, a solution and a scenario of a QSS is the sequence of solutions and scenarios of all its QCNs respectively.

**Definition 7** A QSS  $\mathcal{S}$  is satisfiable iff it admits a solution.

In what follows, we will be interested in studying atomic QSSs, as the results obtained for that case can be carried to the general case of QSSs as well.

**Example.** An example of a spatiotemporal sequence based on RCC-8 is given in Figure 5. Figure 5 depicts the sequence  $(\mathcal{N}_a = (V, C_a), \mathcal{N}_b = (V, C_b), \mathcal{N}_c = (V, C_c), \mathcal{N}_d = (V, C_d), \mathcal{N}_e = (V, C_e), \mathcal{N}_f = (V, C_f))$ , where  $V = \{x, y, z\}$  is its set of variables and  $\mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c, \mathcal{N}_d, \mathcal{N}_e$ , and  $\mathcal{N}_f$  are RCC-8 configurations over  $V$ . In particular,  $\mathcal{N}_a$  defines the set of constraints  $\{DC(x, y), DC(y, z), DC(x, z)\}$ ,  $\mathcal{N}_b$  defines the set of constraints  $\{EC(x, y), DC(y, z), DC(x, z)\}$ ,  $\mathcal{N}_c$  defines the set of constraints  $\{EC(x, y), DC(y, z), EC(x, z)\}$ ,  $\mathcal{N}_d$  defines the set of constraints  $\{PO(x, y), DC(y, z), EC(x, z)\}$ ,  $\mathcal{N}_e$  defines the set of constraints  $\{TPPi(x, y), DC(y, z), EC(x, z)\}$ , and finally  $\mathcal{N}_f$  defines the set of constraints  $\{NTPPi(x, y), DC(y, z), DC(x, z)\}$ . Each spatial QCN in the sequence corresponds to a unique point of time in the timeline  $t$ . For example, spatial configuration  $\mathcal{N}_c$  corresponds to the point of time  $t_c$  in the timeline  $t$ . Thus, the ordering of the spatial QCNs in a given sequence yields a spatiotemporal configuration that describes how a spatial configuration

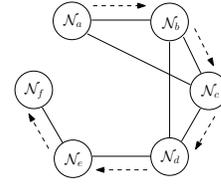


Figure 6: Transition graph of a spatiotemporal sequence

evolves over time.

At this point, we can extend the notion of conceptually neighbouring relations to the notion of conceptually neighbouring atomic QCNs as follows:

**Definition 8** Given a qualitative constraint language  $\mathcal{L}$  and a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , we have that two atomic QCNs  $\mathcal{N} = (V, C)$  and  $\mathcal{N}' = (V, C')$  of  $\mathcal{L}$  are conceptual neighbours with respect to  $\Gamma$  if  $\forall u, v \in V$  we have that  $b(u, v)$  and  $b'(u, v)$  are conceptual neighbours with respect to  $\Gamma$ , where  $b(u, v)$  and  $b'(u, v)$  are the base relations defined by the singleton relations  $C(u, v)$  and  $C'(u, v)$  respectively.

Intuitively, two atomic QCNs are conceptual neighbours if they can transition from one another by simultaneous transformation of their base relations to conceptually neighbouring base relations. We can also give the following definition of a conceptual neighbourhood graph for a set of atomic QCNs, but to avoid any confusion with the conceptual neighbourhood graph of the base relations of a qualitative constraint language  $\mathcal{L}$  we will refer to it as a transition graph<sup>1</sup>:

**Definition 9** Given a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , and a satisfiable atomic QSS  $\mathcal{S} = (\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k)$  of  $\mathcal{L}$ , the transition graph of  $\mathcal{S}$  defined with respect to  $\Gamma$  is the graph  $M = (\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}, E)$  where  $E = \{(\mathcal{N}_i, \mathcal{N}_j) \mid \mathcal{N}_i, \mathcal{N}_j \in \{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}; \text{ and } \mathcal{N}_i \text{ and } \mathcal{N}_j \text{ being conceptual neighbours with respect to } \Gamma\}$ .

The transition graph of a satisfiable atomic QSS  $\mathcal{S}$  of  $k$  QCNs encodes all the conceptually allowed transitions between its spatial QCNs, i.e., it encodes all the pairs of atomic QCNs that are conceptual neighbours with respect to an assumed conceptual neighbourhood graph. Clearly, if the QCNs are defined over a set of variables  $V$ , it takes  $O(|V|^2)$  time to calculate if a transition is possible between the QCNs of a given pair of QCNs. As the transition graph of  $\mathcal{S}$  has  $k$  nodes and, thus,  $O(k^2)$  possible edges, i.e., pairs of QCNs, obtaining the entire transition graph can be done in polynomial time. It is also the case that every node, i.e., every QCN, in a transition graph is a conceptual neighbour of itself.

**Example.** As an example, the transition graph of the spatiotemporal sequence depicted in Figure 5, is shown in Figure 6. Indeed, we can have continuous transitions between the spatial QCNs in the pairs  $(\mathcal{N}_a, \mathcal{N}_b)$ ,  $(\mathcal{N}_b, \mathcal{N}_c)$ ,

<sup>1</sup>In fact, the reader can easily verify that in the case where we have the set of all possible atomic QCNs of  $\mathcal{L}$  over exactly two spatial entities, the transition graph defined by those QCNs corresponds to the conceptual neighbourhood graph of  $\mathcal{L}$ .

$(\mathcal{N}_c, \mathcal{N}_d)$ ,  $(\mathcal{N}_d, \mathcal{N}_e)$ ,  $(\mathcal{N}_e, \mathcal{N}_f)$  of consecutive QCNs in the sequence  $(\mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c, \mathcal{N}_d, \mathcal{N}_e, \mathcal{N}_f)$ , but also continuous transitions between spatial configurations  $\mathcal{N}_a$  and  $\mathcal{N}_c$  (i.e., the pair  $(\mathcal{N}_a, \mathcal{N}_c)$ ), and  $\mathcal{N}_b$  and  $\mathcal{N}_d$  (i.e., the pair  $(\mathcal{N}_b, \mathcal{N}_d)$ ).

Let us recall the definition of a Hamiltonian path.

**Definition 10** ([Garey and Johnson, 1979]) *Given a graph  $G$ , a Hamiltonian path in  $G$  is a path that visits each vertex  $v \in V(G)$  exactly once.*

It is easy to see that the pairs of consecutive QCNs in the sequence of the aforementioned example correspond to a Hamiltonian path illustrated with dashed arrows in Figure 6. Based on this observation, we will now formally introduce the main problem that we are interested in studying in this paper and sketch its relation with the problem of finding a Hamiltonian path in a given graph, which is known to be  $\mathcal{NP}$ -complete [Garey and Johnson, 1979]. In fact, we will make a polynomial-time reduction of the Hamiltonian path problem to our problem. We call our problem the sequence ordering problem (SOP) and define it as follows:

**Definition 11** *Given a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , and a satisfiable atomic QSS  $\mathcal{S} = (\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k)$  of  $\mathcal{L}$ , the SOP for  $\mathcal{S}$  is the problem of obtaining an ordered sequence of the QCNs of  $\mathcal{S}$  such that the spatial QCNs  $\mathcal{N}_i$  and  $\mathcal{N}_j$  in every pair of consecutive QCNs  $(\mathcal{N}_i, \mathcal{N}_j)$  in the ordered sequence are conceptual neighbours with respect to  $\Gamma$ .*

The relation between the Hamiltonian path problem and the SOP is as follows:

**Lemma 1** *Given a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , a satisfiable atomic QSS  $\mathcal{S}$  of  $\mathcal{L}$ , and the transition graph  $M$  of  $\mathcal{S}$  defined with respect to  $\Gamma$ , solving the SOP for  $\mathcal{S}$  is equivalent to obtaining a Hamiltonian path in  $M$ .*

Indeed, as Lemma 1 suggests, a Hamiltonian path in the transition graph of a given qualitative spatiotemporal sequence, will provide us with an ordered sequence of its QCNs such that the QCNs in every pair of consecutive QCNs in the ordered sequence are conceptual neighbours with respect to some conceptual neighbourhood graph, and vice versa, as explained earlier in light of our example.

We provide a definition on graph isomorphism that will be of use in what follows.

**Definition 12** *A graph  $G_1 = (V_1, E_1)$  is isomorphic to a graph  $G_2 = (V_2, E_2)$  iff there is a bijection  $f : V_1 \rightarrow V_2$  such that  $(u, v) \in E_1$  iff  $(f(u), f(v)) \in E_2$ .*

It might be tempting at this point to suggest that the SOP for a given QSS  $\mathcal{S}$  is  $\mathcal{NP}$ -complete, as is the case with the Hamiltonian path problem. However, we first need to show that any arbitrary graph  $G$  can be translated to an isomorphic to  $G$  transition graph  $M$  in polynomial time. This is a necessary requirement in our line of reasoning for proving  $\mathcal{NP}$ -completeness for the SOP, as it could be the case that for a given qualitative constraint language and its conceptual neighbourhood graph, the family of transition graphs that can be constructed allow for obtaining a Hamiltonian path in

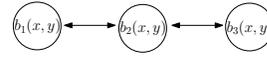


Figure 7: A conceptual neighbourhood graph

polynomial time. A trivial case, for example, would be knowing for a fact that any transition graph is a complete graph. Further, to be able to prove  $\mathcal{NP}$ -completeness for the SOP, we require that a qualitative constraint language has the  $P_3$  property, defined as follows:

**Property 1 (Property  $P_3$ )** *A qualitative constraint language  $\mathcal{L}$  will be said to have the  $P_3$  property if it satisfies the following conditions:*

- $B_{\mathcal{L}}$  consists of at least three base relations  $b_1$ ,  $b_2$ , and  $b_3$ ;
- The conceptual neighbourhood graph defined by base relations  $b_1$ ,  $b_2$ , and  $b_3$  is the graph  $\Gamma = (\{b_1(u, v), b_2(u, v), b_3(u, v)\}, \{(b_1(u, v), b_2(u, v)), (b_2(u, v), b_3(u, v))\})$ , with  $u$  and  $v$  being two entities, as shown in Figure 7 (omitting loops);
- Base relation  $b_1$  belongs to all the possible weak compositions among base relations  $b_1$ ,  $b_2$ , and  $b_3$ , viz.,  $b_1 \in b_i \diamond b_j \forall i, j \in \{1, 2, 3\}$ ;
- The satisfiability of an atomic QCN defined by base relations  $b_1$ ,  $b_2$ , and  $b_3$  can be decided by  $\diamond$ -consistency.

By considering the base relations  $DC$  (disconnected),  $EC$  (externally connected), and  $PO$  (partially overlaps) for RCC-8 [Randell et al., 1992], the base relations  $<$  (before),  $m$  (meets), and  $o$  (overlaps) for IA [Allen, 1983], and the base relations  $<$  (left of),  $\leq$  (attached to), and  $\Leftarrow$  (overlapping) for RA [Guesgen, 1989], we can obtain the following proposition:

**Proposition 1** *The qualitative constraint languages RCC-8, IA, and RA have the  $P_3$  property.*

Let us go back to being able to construct a transition graph out of any given arbitrary graph in polynomial time. We prove the following proposition:

**Proposition 2** *Given a graph  $G$ , and a qualitative constraint language  $\mathcal{L}$  having the  $P_3$  property, we have that we can construct a satisfiable atomic QSS  $\mathcal{S}$  of  $\mathcal{L}$  that yields an isomorphic to  $G$  transition graph  $M$  in polynomial time.*

**Proof.** Given an arbitrary graph  $G = (V, E)$ , and a qualitative constraint language  $\mathcal{L}$  that has the  $P_3$  property, we can construct a set of satisfiable atomic QCNs of  $\mathcal{L}$  that yield a transition graph which is isomorphic to  $G$  using algorithm Arachni, depicted in Algorithm 1. We prove the correctness of Arachni as follows. If the order of graph  $G$  is  $k$ , i.e., if  $k = |V|$ , we create a set  $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$  of  $k$  QCNs of  $\mathcal{L}$ . In fact, we have a bijection between sets  $V$  and  $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$ , as we consider to have a one-to-one correspondence between an element of  $V$  and a QCN in the set of  $k$  QCNs of  $\mathcal{L}$ . This bijection is defined by a dictionary map that given a node  $v \in V$  returns the index  $i$  of a QCN  $\mathcal{N}_i$  in the set of  $k$  QCNs of  $\mathcal{L}$ , i.e.,  $i = \text{map}[v]$ , with  $i \in \{1, 2, \dots, k\}$ . For every  $i \in \{1, 2, \dots, k\}$ , we have that every QCN  $\mathcal{N}_i$  shares the set of variables  $\{v_1, v_2, \dots, v_{k+1}\}$ , viz., all  $k$  QCNs of  $\mathcal{L}$  are defined over the same set of variables  $\{v_1, v_2, \dots, v_{k+1}\}$ .

**Algorithm 1:** Arachni( $G, \mathcal{L}$ )

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in      : A graph  $G = (V, E)$ , and a qualitative constraint
           language  $\mathcal{L}$  that has the  $P_3$  property.
output  : A set of satisfiable atomic QCNs of  $\mathcal{L}$  that yield a
           transition graph which is isomorphic to graph  $G$ .

1 begin
2    $i \leftarrow 1$ ;
3    $\chi \leftarrow \emptyset$ ;
4    $map \leftarrow \text{dict}()$ ;
5   while  $V$  do
6      $map[V.\text{pop}()] \leftarrow i$ ;
7      $V_i \leftarrow \{v_1, v_2, \dots, v_{|V(G)|+1}\}$ ;
8     foreach  $v_k, v_l \in V_i$  do
9       if  $k = l$  then
10         $C_i(v_k, v_l) \leftarrow \{\text{Id}\}$ ;
11       else if  $k < l$  then
12         if  $k = i$  and  $l = k + 1$  then
13            $C_i(v_k, v_l) \leftarrow \{b_3\}$ ;
14            $C_i(v_l, v_k) \leftarrow \{b_3^{-1}\}$ ;
15         else
16            $C_i(v_k, v_l) \leftarrow \{b_1\}$ ;
17            $C_i(v_l, v_k) \leftarrow \{b_1^{-1}\}$ ;
18        $\mathcal{N}_i \leftarrow (V_i, C_i)$ ;
19        $\chi \leftarrow \chi \cup \{\mathcal{N}_i\}$ ;
20        $i \leftarrow i + 1$ ;
21   while  $E$  do
22      $(u, u') \leftarrow E.\text{pop}()$ ;
23      $(i, j) \leftarrow (map[u], map[u'])$ ;
24      $\mathcal{N}_i[v_j, v_{j+1}] \leftarrow \{b_2\}$ ;  $\mathcal{N}_i[v_{j+1}, v_j] \leftarrow \{b_2^{-1}\}$ ;
25      $\mathcal{N}_j[v_i, v_{i+1}] \leftarrow \{b_2\}$ ;  $\mathcal{N}_j[v_{i+1}, v_i] \leftarrow \{b_2^{-1}\}$ ;
26   return  $\chi$ ;
```

---

We assume first that  $G$  is an edgeless graph, therefore, the  $k$  QCNs of  $\mathcal{L}$  are initially constructed in a manner such that there exists no pair of QCNs where the QCNs in the pair are conceptual neighbours of one another. This is achieved by initializing relation  $\mathcal{N}_i[v_i, v_{i+1}]$  for every QCN  $\mathcal{N}_i$  with the singleton relation  $\{b_3\}$  while initializing all other relations  $\mathcal{N}_i[v_j, v_o]$  with  $i \neq j$  and  $j < o$  with the singleton relation  $\{b_1\}$ . Then, for any pair of QCNs ( $\mathcal{N}_i, \mathcal{N}_j$ ) from our set of  $k$  QCNs of  $\mathcal{L}$ , we have that  $\mathcal{N}_i$  and  $\mathcal{N}_j$  are not conceptual neighbours, since the base relations  $b_3$  and  $b_1$  defined by relations  $\mathcal{N}_i[v_i, v_{i+1}]$  and  $\mathcal{N}_j[v_i, v_{i+1}]$  respectively (and equivalently, the base relations  $b_1$  and  $b_3$  defined by relations  $\mathcal{N}_i[v_j, v_{j+1}]$  and  $\mathcal{N}_j[v_j, v_{j+1}]$  respectively) are not conceptual neighbours. Up to this point it should be clear that we have constructed a set of atomic QCNs that yield a transition graph which is isomorphic to an edgeless graph of order  $k$ . Since every QCN in our set of  $k$  QCNs of  $\mathcal{L}$  is defined over  $k + 1$  entities, and assuming that we use a matrix to represent a given QCN, the construction of our QCNs is achieved in  $O(k^3)$  time. Now, we need to iterate the set of edges of graph  $G$  and change the QCNs in the corresponding pairs of QCNs into being conceptual neighbours of one another. Using dictionary map, we obtain a pair of QCNs ( $\mathcal{N}_i, \mathcal{N}_j$ ) for every edge  $(u, u') \in E$ , where  $i = \text{map}[u]$  and  $j = \text{map}[u']$ . As noted earlier,  $\mathcal{N}_i$  and  $\mathcal{N}_j$  are not conceptual neighbours, since the base relations  $b_3$  and  $b_1$  defined by relations  $\mathcal{N}_i[v_i, v_{i+1}]$  and  $\mathcal{N}_j[v_i, v_{i+1}]$  re-

spectively (and equivalently, the base relations  $b_1$  and  $b_3$  defined by relations  $\mathcal{N}_i[v_j, v_{j+1}]$  and  $\mathcal{N}_j[v_j, v_{j+1}]$  respectively) are not conceptual neighbours. Therefore, we need to change the aforementioned base relations  $b_1$  into being base relation  $b_2$ , so that we can achieve conceptual proximity with base relation  $b_3$ . In particular, we set relations  $\mathcal{N}_j[v_i, v_{i+1}]$  and  $\mathcal{N}_i[v_j, v_{j+1}]$  to  $\{b_2\}$  from  $\{b_1\}$ . Note that QCNs  $\mathcal{N}_i$  and  $\mathcal{N}_j$  become conceptual neighbours only of one another, as any other QCN  $\mathcal{N}_o$  with  $i \neq o \neq j$  is not a conceptual neighbour to either  $\mathcal{N}_i$  or  $\mathcal{N}_j$ , since relation  $\mathcal{N}_o[v_o, v_{o+1}]$  is defined by  $b_3$ , and relations  $\mathcal{N}_i[v_o, v_{o+1}]$  and  $\mathcal{N}_j[v_o, v_{o+1}]$  are still defined by  $b_1$  (and equivalently, relations  $\mathcal{N}_o[v_i, v_{i+1}]$  and  $\mathcal{N}_o[v_j, v_{j+1}]$  are defined by  $b_1$ , and relations  $\mathcal{N}_i[v_i, v_{i+1}]$  and  $\mathcal{N}_j[v_j, v_{j+1}]$  are defined by  $b_3$ ). After iterating the whole set of edges of graph  $G$ , we will have that any two nodes  $u$  and  $u'$  of  $G$  are adjacent in  $G$  if and only if  $\mathcal{N}_{\text{map}[u]}$  and  $\mathcal{N}_{\text{map}[u']}$  are adjacent in the transition graph that is defined by the set  $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$  of  $k$  QCNs of  $\mathcal{L}$ . Formally, if  $M$  is the transition graph defined by the set  $\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}$  of  $k$  QCNs of  $\mathcal{L}$ , we have that  $(u, u') \in E(G)$  iff  $(\mathcal{N}_{\text{map}[u]}, \mathcal{N}_{\text{map}[u']}) \in E(M)$ . Thus, graph  $M$  is isomorphic to graph  $G$ . To fix the pairs of QCNs that are conceptual neighbours and consequently introduce the edges in our transition graph, we require  $O(k^2)$  time, as there can only be  $O(k^2)$  edges in a  $k$  order graph (and given that our QCNs are represented by matrices we can alter their relations in  $O(1)$  time). In conclusion, algorithm Arachni requires  $O(k^3)$  running time in total to process its input and produce an output. Finally, we also need to show that every QCN in the set of  $k$  atomic QCNs of  $\mathcal{L}$  that we have constructed is satisfiable. Due to our construction, for every QCN  $\mathcal{N}_i = (V_i, C_i)$ , with  $i \in \{1, 2, \dots, k\}$ , we have that every triple of variables  $v_o, v'_o$  and  $v''_o$  in  $V_i$ , with  $o < o' < o''$ , defines a set of relations  $\mathcal{N}_i[v_o, v_{o'}]$ ,  $\mathcal{N}_i[v_{o'}, v_{o''}]$ , and  $\mathcal{N}_i[v_o, v_{o''}]$ , such that  $\mathcal{N}_i[v_o, v_{o''}]$  is always defined by the base relation  $b_1$ , and  $\mathcal{N}_i[v_o, v_{o'}]$  and  $\mathcal{N}_i[v_{o'}, v_{o''}]$  are defined by either of the three base relations  $b_1, b_2$ , and  $b_3$ . Due to the fact that  $b_1 \in b_i \diamond b_j \forall i, j \in \{1, 2, 3\}$ , we have that  $\mathcal{N}_i[v_o, v_{o''}] \subseteq \mathcal{N}_i[v_o, v_{o'}] \diamond \mathcal{N}_i[v_{o'}, v_{o''}]$ . Further, as  $\mathcal{L}$  is a relation algebra and therefore satisfies the axioms of  $\diamond$ -associativity,  $^{-1}$ -involution,  $^{-1}$ -involutive distributivity, and Peircean law (sometimes called cycle law) [Dylla *et al.*, 2013], we can decide that every path of length 2 in  $\mathcal{N}_i$  is closed under the weak composition operation defined by operator  $\diamond$ , thus,  $\mathcal{N}_i$  is  $\diamond$ -consistent. As  $\diamond$ -consistency decides the satisfiability of atomic QCNs of  $\mathcal{L}$ , we have that  $\mathcal{N}_i$  is a satisfiable QCN of  $\mathcal{L}$  for every  $i \in \{1, 2, \dots, k\}$ .  $\dashv$

We proceed with obtaining a complexity result for the SOP, for the case where the considered satisfiable atomic QSS  $\mathcal{S}$  is defined over a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ .

**Theorem 1** *The SOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ , is  $\mathcal{NP}$ -complete.*

**Proof.**  $\mathcal{NP}$ -hardness follows from the fact that the Hamiltonian path problem is  $\mathcal{NP}$ -complete, and we can translate any input of the Hamiltonian path problem, which is an arbitrary graph  $G$ , to an isomorphic to  $G$  transition graph  $M$  of some

QSS  $\mathcal{S}$  in polynomial time, due to Proposition 2. Further, due to the notion of isomorphism, it is clear that we can have a Hamiltonian path in  $M$  iff we can have a Hamiltonian path in  $G$ . By Lemma 1, we have that obtaining a Hamiltonian path in  $M$  is equivalent to solving the SOP for  $\mathcal{S}$ , thus, we ultimately have obtained a polynomial-time reduction from the Hamiltonian path problem to the SOP. We can also explicitly define membership in  $\mathcal{NP}$  due to the fact that provided with a candidate ordered satisfiable atomic QSS  $\mathcal{S}$ , we can check if the QCNs in every pair of consecutive QCNs in  $\mathcal{S}$  are conceptual neighbours in polynomial time. In particular, if  $\mathcal{S}$  comprises  $k$  QCNs, we can only have  $k - 1$  pairs of consecutive QCNs in the sequence, and we can check if the QCNs in a pair are conceptual neighbours in  $O(n^2)$  time, given the fact that the QCNs are defined over  $n$  entities. (Note also that as suggested in the proof of Proposition 2,  $n$  can be equal to  $k + 1$ ). Thus, the SOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ , is  $\mathcal{NP}$ -complete.  $\dashv$

Due to Proposition 1, we can obtain the following result:

**Corollary 1** *The SOP for any satisfiable atomic QSS  $\mathcal{S}$  of RCC-8, IA, or RA, is  $\mathcal{NP}$ -complete.*

We can obtain a variation of the SOP for a satisfiable atomic QSS  $\mathcal{S}$ , where we allow one to consider a number of up to  $m$  QCNs in addition to the number of QCNs of  $\mathcal{S}$  and solve the SOP for the new augmented QSS  $\mathcal{S}'$ . This is particularly useful if given a QSS  $\mathcal{S}$  we are unable to solve the SOP for  $\mathcal{S}$ , because  $\mathcal{S}$ , for example, yields a disconnected transition graph and, thus, does not allow obtaining a Hamiltonian path in its transition graph. We provide a simple, but, nevertheless, sufficient example to better explain this problem.

**Example.** Let RCC-8 be our qualitative constraint language of choice with its usual conceptual neighbourhood graph as depicted in Figure 4, and  $(\mathcal{N}_a, \mathcal{N}_b)$  a QSS  $\mathcal{S}$  of RCC-8, where  $\mathcal{N}_a$  defines the set of constraints  $\{DC(x, y)\}$  and  $\mathcal{N}_b$  defines the set of constraints  $\{PO(x, y)\}$ . Clearly, the transition graph of  $\mathcal{S}$  is disconnected as  $\mathcal{N}_a$  and  $\mathcal{N}_b$  are not conceptual neighbours and, thus, there can be no transition from  $\mathcal{N}_a$  to  $\mathcal{N}_b$ , and vice versa. In particular, the transition graph of  $\mathcal{S}$  is the graph  $M = (\{\mathcal{N}_a, \mathcal{N}_b\}, \emptyset)$ . As such, the SOP for  $\mathcal{S}$  is unsolvable, since there can be no Hamiltonian path in  $M$ . However, we can augment  $\mathcal{S}$  with the QCN  $\mathcal{N}_c$  that defines the set of constraints  $\{EC(x, y)\}$ , and obtain the QSS  $\mathcal{S}' = (\mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c)$ . Then, the transition graph of  $\mathcal{S}'$  will be the graph  $M' = (\{\mathcal{N}_a, \mathcal{N}_b, \mathcal{N}_c\}, \{(\mathcal{N}_a, \mathcal{N}_b), (\mathcal{N}_b, \mathcal{N}_c)\})$ . The Hamiltonian path  $(\mathcal{N}_a, \mathcal{N}_c, \mathcal{N}_b)$  in  $M'$  is exactly a solution of the SOP for  $\mathcal{S}'$ , where we considered one extra QCN with respect to the number of QCNs of  $\mathcal{S}$ .

We call this new problem the *relaxed* sequence ordering problem (rSOP) and define it as follows:

**Definition 13** *Given an integer  $m$ , a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , and a satisfiable atomic QSS  $\mathcal{S} = (\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k)$  of  $\mathcal{L}$  over a set of variables  $V$ , the rSOP for  $\mathcal{S}$  is the SOP for QSS  $\mathcal{S}'$ , where  $\mathcal{S}'$  is the sequence  $\mathcal{S}$  augmented with a set  $\{\mathcal{N}'_1, \mathcal{N}'_2, \dots, \mathcal{N}'_n\}$  of  $n$  QCNs of  $\mathcal{L}$  over  $V$ , with  $n \leq m$ .*

We proceed with obtaining a complexity result for the rSOP, for the case where the considered satisfiable atomic QSS  $\mathcal{S}$  is defined over a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ .

**Theorem 2** *The rSOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$  and some integer  $m$ , is  $\mathcal{NP}$ -complete.*

**Proof.**  $\mathcal{NP}$ -hardness follows from the fact that the SOP, which is  $\mathcal{NP}$ -complete due to Theorem 1, can be reduced to the rSOP in polynomial time, by just considering an integer value of  $m = 0$  for the rSOP. With the aforementioned requirement for integer  $m$ , it is clear that any input for the SOP serves as an input for the rSOP and a solution of the rSOP is also a solution of the SOP, and vice versa. Membership in  $\mathcal{NP}$  follows from the fact that provided with a candidate ordered satisfiable atomic QSS  $\mathcal{S}'$  which corresponds to an input satisfiable atomic QSS  $\mathcal{S}$  augmented with  $\leq m$  QCNs, we can check if  $\mathcal{S}'$  is a solution of the SOP for  $\mathcal{S}'$  in polynomial time, as the SOP is in  $\mathcal{NP}$ . Also, we can check if  $\mathcal{S}'$  contains  $\leq m$  more QCNs than  $\mathcal{S}$  in linear time in the number of QCNs of  $\mathcal{S}'$ . Thus, the rSOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$  and some integer  $m$ , is  $\mathcal{NP}$ -complete.  $\dashv$

Due to Proposition 1, we can obtain the following result:

**Corollary 2** *The rSOP for any satisfiable atomic QSS  $\mathcal{S}$  of RCC-8, IA, or RA and some integer  $m$ , is  $\mathcal{NP}$ -complete.*

The rSOP, as is the case with the SOP, is a decision problem where we try to decide if an adequate ordered sequence exists, and if so, present that sequence as a solution of some input instance. However, we can also view the rSOP as an *optimization problem* [Krentel, 1988; Creignou *et al.*, 2001] where we try to minimize the integer value of  $m$ .

Before closing this section, let us introduce yet another problem that deals with *digraphs* (also called directed graphs). We can view the transition graph of a satisfiable atomic QSS as a digraph, where the edges, i.e., the pairs of QCNs, have a direction associated with them that specifies which QCN in the pair can transition to the other one. We call the corresponding problem the *directed* sequence ordering problem (dSOP) and define it as follows:

**Definition 14** *Given a qualitative constraint language  $\mathcal{L}$ , a conceptual neighbourhood graph  $\Gamma$  of  $\mathcal{L}$ , a satisfiable atomic QSS  $\mathcal{S} = (\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k)$  of  $\mathcal{L}$ , and a transition digraph  $M_d = (\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}, A)$ , with  $A = \{(\mathcal{N}_i, \mathcal{N}_j) \text{ and/or } (\mathcal{N}_j, \mathcal{N}_i) \mid (\mathcal{N}_i, \mathcal{N}_j) \in E\}$ , where  $M = (\{\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k\}, E)$  is the transition graph of  $\mathcal{S}$  defined with respect to  $\Gamma$ , the dSOP for  $\mathcal{S}$  is the problem of obtaining an ordered sequence of the QCNs of  $\mathcal{S}$  such that the spatial QCNs  $\mathcal{N}_i$  and  $\mathcal{N}_j$  in every pair of consecutive QCNs  $(\mathcal{N}_i, \mathcal{N}_j)$  in the ordered sequence are conceptual neighbours with respect to  $\Gamma$  and  $(\mathcal{N}_i, \mathcal{N}_j) \in A$ .*

We proceed with obtaining a complexity result for the dSOP, for the case where the considered satisfiable atomic QSS  $\mathcal{S}$  is defined over a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ .

**Theorem 3** *The dSOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ , is  $\mathcal{NP}$ -complete.*

**Proof.**  $\mathcal{NP}$ -hardness follows from the fact that the SOP, which is  $\mathcal{NP}$ -complete due to Theorem 1, can be reduced to the dSOP in polynomial time, by just considering a transition digraph  $M_d = (V, A)$  of  $\mathcal{S}$ , with  $A = \{(\mathcal{N}_i, \mathcal{N}_j) \text{ and } (\mathcal{N}_j, \mathcal{N}_i) \mid (\mathcal{N}_i, \mathcal{N}_j) \in E\}$ , where  $M = (V, E)$  is the transition graph of  $\mathcal{S}$  defined with respect to  $\Gamma$ . Namely, for every edge in  $M$  we introduce both directions of this edge, i.e., both arcs, in  $M_d$ . With the aforementioned requirement for the transition digraph  $M_d$ , it is clear that any input for the SOP serves as an input for the dSOP and a solution of the dSOP is also a solution of the SOP, and vice versa. Membership in  $\mathcal{NP}$  follows from the fact that provided with a candidate ordered satisfiable atomic QSS  $\mathcal{S}$ , we need to check if  $\mathcal{S}$  is a solution of the SOP for  $\mathcal{S}$  and also check if the QCNs in every pair of  $k - 1$  pairs of consecutive QCNs in  $\mathcal{S}$  form an arc that belongs to the transition digraph  $M_d$ . We can perform the former check in polynomial time as the SOP is in  $\mathcal{NP}$ . For the latter check, if we assume that we use a matrix to store the transition digraph  $M_d$ , we can check if a pair of QCNs forms an arc that belongs to the transition digraph  $M_d$  in  $O(1)$  time, thus, we need  $O(k - 1)$  time in total for all  $k - 1$  pairs of QCNs. As such, the dSOP for any satisfiable atomic QSS  $\mathcal{S}$  of a qualitative constraint language  $\mathcal{L}$  satisfying property  $P_3$ , is  $\mathcal{NP}$ -complete.  $\dashv$

Due to Proposition 1, we can obtain the following result:

**Corollary 3** *The dSOP for any satisfiable atomic QSS  $\mathcal{S}$  of RCC-8, IA, or RA, is  $\mathcal{NP}$ -complete.*

## 4 Conclusion and future work

In this paper, we investigated the task of ordering a temporal sequence of qualitative spatial configurations, where specific transition constraints with respect to a conceptual neighbourhood graph of a qualitative spatial constraint language are assumed. In particular, we showed that the problem of ordering a sequence of qualitative spatial configurations to meet such transition constraints is  $\mathcal{NP}$ -complete for the well known languages of RCC-8, Interval Algebra, and Rectangle Algebra. As our results lie within the area of Graph Traversal, they allow for many practical and diverse applications, such as identifying optimal routes in mobile robot navigation, modelling changes of topology in biological processes, and computing sequences of segmentation steps used in image processing algorithms. A direct consequence of our work would be to generalize to tree and graph structures to capture the temporal aspect of qualitative spatial configurations.

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