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Studying the Use and Effect of Graph Decomposition in Qualitative Spatial and Temporal Reasoning

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Abstract

We survey the use and effect of decomposition-based techniques in qualitative spatial and temporal constraint-based reasoning, and clarify the notions of a tree decomposition, a chordal graph, and a partitioning graph, and their implication with a particular constraint property that has been extensively used in the literature, namely, patchwork. As a consequence, we prove that a recently proposed decomposition-based approach that was presented in (Nikolaou et al., "Fast Consistency Checking of Very Large Real-World RCC-8 Constraint Networks Using Graph Partitioning", in *AAAI*, 2014) for checking the satisfiability of qualitative spatial constraint networks lacks soundness. Therefore, the approach becomes quite controversial as it does not seem to offer any technical advance at all, while experimental evaluation of it in a following work presented in (Sioutis, "Triangulation versus Graph Partitioning for Tackling Large Real World Qualitative Spatial Networks", in *ICTAI*, 2014) becomes questionable. Finally, we present a particular tree decomposition that is based on the biconnected components of the constraint graph of a given large network, and show that it allows for cost-free utilization of parallelism for a qualitative constraint language that has patchwork for satisfiable atomic networks.

1 Introduction

Qualitative Spatial and Temporal Reasoning (QSTR) is a major field of study in Artificial Intelligence and, particularly, in Knowledge Representation. This field studies representations of space and time that abstract from numerical quantities. The concise expressiveness of the representational languages used in the qualitative approach provides a promising framework that boosts research and applications in a plethora of areas and domains, such as ambient intelligence, dynamic GIS, cognitive robotics, and spatiotemporal design (Hazarika 2012; Bhatt et al. 2011).

The Interval Algebra (IA) (Allen 1981) and a fragment of the Region Connection Calculus (RCC) (Randell, Cui, and Cohn 1992), namely RCC-8, are the dominant Artificial Intelligence approaches for representing and reasoning about qualitative temporal and topological relations respectively. These qualitative calculi use constraints to encode knowledge about the spatial or temporal relationships between entities in an abstract manner. Thus, the problem of reasoning about qualitative information can be modeled as an infinite-domain variant of a Constraint Satisfaction Problem (CSP) (Montanari 1974), for which we use the term *Qualitative Constraint Network* (QCN). For instance, there are infinitely many time points or temporal intervals in the timeline and infinitely many regions in a two- or three-dimensional space. One way of dealing with infinite domains is using constraints over a finite set of binary relations, called *base relations* (or *atoms*), by employing a *relation algebra* (Ladkin and Maddux 1994).

Given a QCN over a set of variables corresponding to a set of spatial or temporal entities, we are particularly interested in its *satisfiability problem*, that is, deciding whether there exists an interpretation of all the variables of the QCN such that all of its constraints are satisfied by this interpretation; such an interpretation being called a *solution*. The satisfiability problem is closely related to the *minimal labeling problem* (MLP) (Amaneddine, Condotta, and Sioutis 2013; Liu and Li 2012) and the *redundancy problem* (Duckham et al. 2014; Li et al. 2015), in the sense that the latter two problems exhibit functions that build on the core algorithms used to obtain a solution of a QCN. In particular, the MLP is the problem of determining all the base relations for each of the constraints of a QCN that participate in at least one solution of the QCN, whilst the redundancy problem is the problem of obtaining all the constraints of a QCN that do not contain at least one base relation participating in a solution of the modified QCN that results by removing these constraints (these constraints being referred to as non-redundant constraints, since their removal changes the solution set of the QCN). As such, we will emphasize on the satisfiability problem throughout the paper, and make the link whenever there is a work related to the MLP or the redundancy problem whose results are affected by the notions and techniques that we will survey here. The satisfiability problem in IA and RCC-8 is $\mathcal{NP}$-complete in general (Renz and Nebel 1999; Nebel and Bürckert 1995). However, there exist large maximal tractable subclasses of relations of IA and RCC-8, which can be used to make reasoning much more efficient even in the general $\mathcal{NP}$-complete case (Nebel 1997; Renz and Nebel 2001). In recent years, many works surfaced that use graph decomposition to significantly improve the efficiency and scalability of practical reasoning (Sioutis and Condotta 2014; Amaneddine, Condotta, and Sioutis 2013; Sioutis and Koubarakis 2012; Condotta and D'Almeida 2011; Sioutis, Condotta, and Koubarakis 2015; Li, Huang, and Renz 2009; Huang, Li, and Renz 2013; Nikolaou and Koubarakis 2014; Chmeiss and Condotta 2011; Westphal, Hué, and Wölfl 2013; Westphal and Hué 2014). All these works, make use of a particular constraint property, namely, patchwork (Lutz and Milicic 2007; Huang 2012). Intuitively, patchwork ensures that the combination of two satisfiable QCNs that completely agree on the constraints between their common variables continues to be satisfiable.

The contribution of this paper is three-fold:

1. we recall the notions of a tree decomposition, a chordal graph, and a partitioning graph that have been used in the literature, and clarify the relationship between one another, and also their implication with patchwork;
2. consequently, we show that the approach proposed in (Nikolaou and Koubarakis 2014) for efficiently checking the satisfiability of qualitative spatial constraint networks violates patchwork in two ways, namely, both in the complete agreement between two satisfiable QCNs and in the graph decomposition that is obtained, and, therefore, lacks soundness;
3. finally, we present a particular tree decomposition that is based on the biconnected components of the constraint graph of a given large QCN, and show that it allows for sound and cost-free utilization of parallelism for a qualitative constraint language that has patchwork for satisfiable atomic QCNs and that it can significantly decongest search when using it for solving non-tractable QCNs.[1]

As such, our paper can be viewed as a survey on the use and effect of graph decomposition in qualitative spatial and temporal reasoning, as a response paper to (Nikolaou and Koubarakis 2014), and partially to (Sioutis 2014), and also as a report of an original decomposition approach that paves the way for efficient utilization of parallelism.

The paper is organised as follows. In Section 2 we recall the definition of a QCN, along with the definition of the property of patchwork. Section 3 introduces the notions of a tree decomposition and a chordal graph, and the way they are interrelated and used in the literature. In Section 4 we present the definition of a partitioning graph (Nikolaou and Koubarakis 2014), and prove that it yields non-soundness when used solely with patchwork. In Section 5 we introduce a particular tree decomposition that allows for cost-free utilization of parallelism for a qualitative constraint language that has patchwork for satisfiable atomic QCNs. In Section 6 we briefly describe QCNs

[1]In whats follows, a tractable (resp. non-tractable) QCN will be a QCN whose satisfiability problem is tractable (resp. non-tractable).

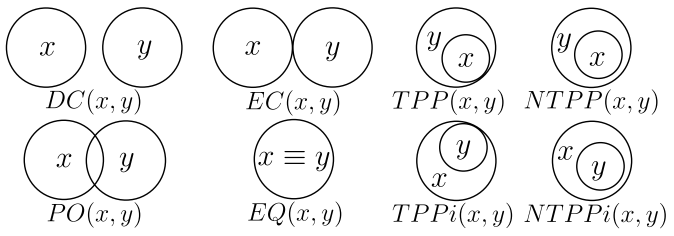


Figure 1 The base relations of the RCC-8 constraint language

in the context of CSPs, and review some related work in the constraint programming community that could inspire future research in the qualitative spatial and temporal reasoning community. Finally, in Section 7 we make a discussion and conclude our work.

2 Preliminaries

A (binary) qualitative temporal or spatial constraint language is based on a finite set B of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain D (Ladkin and Maddux 1994), called the set of base relations. The base relations of the set B of a particular qualitative constraint language can be used to represent the definite knowledge between any two entities with respect to the given level of granularity. B contains the identity relation Id, and is closed under the converse operation ($^{-1}$). Indefinite knowledge can be specified by a union of possible base relations, and is represented by the set containing them. Hence, 2^{B} represents the total set of relations. 2^{B} is equipped with the usual set-theoretic operations (union and intersection), the converse operation, and the weak composition operation denoted by $\diamond$ (Renz and Ligozat 2005).

The set of base relations of RCC-8 (Randell, Cui, and Cohn 1992) is the set {*dc*, *ec*, *po*, *tpp*, *ntpp*, *tppi*, *ntppi*, *eq*}. These eight relations represent the binary topological relations between *regions* that are non-empty regular closed subsets of some topological space, i.e., for any spatial region X we have that $X = c(i(X))$, where $i(\cdot)$ specifies the topological interior of a spatial region and $c(\cdot)$ the topological closure (Renz 2002a). The eight base relations of RCC-8 are depicted in Figure 1 (for the two-dimensional case). The set of base relations of IA (Allen 1981) is the set {*eq*, *p*, *pi*, *m*, *mi*, *o*, *oi*, *s*, *si*, *d*, *di*, *f*, *fi*}. These thirteen relations represent the possible relations between *time intervals*, as depicted in Figure 2.

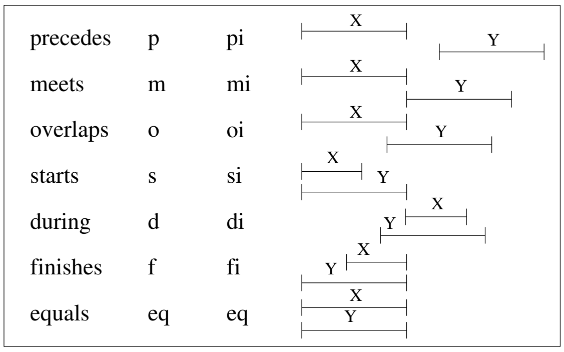
Networks of IA and RCC-8 can be modeled as particular instances of qualitative constraint networks (QCNs), with relation *eq* being the identity relation Id in both cases. A qualitative constraint network is formally defined as follows:

Definition 1 *A* qualitative constraint network (QCN) *is a tuple* (V, C) *where:*

- $V = \{v_1, \ldots, v_n\}$ *is a non-empty finite set of variables corresponding to a set of spatial or temporal entities;*
- C *is a mapping that associates a relation* $r \in 2^{\mathsf{B}}$ *with each pair* (v, v') *of* $V \times V$*, that relation being denoted by* $C(v, v')$*. Further, mapping* C *is such that* $C(v, v) = \{\mathsf{Id}\}$ *and* $C(v, v') = (C(v', v))^{-1}$ *for every* $v, v' \in V$*.*

Note that we always regard a QCN as a complete network. In what follows, given a QCN $\mathcal{N} = (V, C)$ and $v, v' \in V$, the relation $C(v, v')$ will sometimes be denoted by $C_{vv'}$ for simplicity. The constraint graph of a QCN $\mathcal{N} = (V, C)$ is the graph (V, E), denoted by $\mathsf{G}(\mathcal{N})$, for which we have that $(v, v') \in E$ iff $C(v, v') \neq \mathsf{B}$ and $v \neq v'$. (B corresponds to the universal relation, i.e., the non-restrictive[2] relation that contains all base relations, thus, it does not really pose a constraint.) Given a QCN $\mathcal{N} = (V, C)$, $\mathcal{N}$ is said to be *trivially inconsistent* iff $\exists v, v' \in V$ with $C(v, v') = \emptyset$. Further, $\mathcal{N}{\downarrow}_{V'}$, with $V' \subseteq V$, is the QCN $\mathcal{N}$ restricted to V'. A *solution* of $\mathcal{N}$ is a mapping σ

[2]The result of the weak composition between any relation and the universal relation is the universal relation.



precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

Figure 2 The base relations of the Interval Algebra constraint language

defined from V to the domain D, yielding a valid configuration, such that $\forall v, v' \in V$ we have that $(\sigma(v), \sigma(v'))$ can be described by $C(v, v')$, i.e., there exists a base relation $b \in C(v, v')$ such that $(\sigma(v), \sigma(v')) \in b$, viz., $(\sigma(v), \sigma(v'))$ satisfies base relation b.

Definition 2 *A* QCN $\mathcal{N}$ *is* satisfiable *iff it admits a solution. The* satisfiability problem *is the problem of determining if* $\mathcal{N}$ *is satisfiable.*

A *sub*-QCN[3] $\mathcal{N}'$ of $\mathcal{N} = (V, C)$, denoted by $\mathcal{N}' \subseteq \mathcal{N}$, is a QCN (V, C') such that $C'(v, v') \subseteq C(v, v')$ $\forall v, v' \in V$. If b is a base relation, $\{b\}$ is a singleton relation. An *atomic* QCN is a QCN where each constraint is a singleton relation. A *scenario* $\mathcal{S}$ of $\mathcal{N}$ is a satisfiable atomic sub-QCN of $\mathcal{N}$. Given a QCN $\mathcal{N} = (V, C)$, a base relation $b \in C(v, v')$, with $v, v' \in V$, is *feasible* (resp. *unfeasible*) iff there exists (resp. there does not exist) a scenario $\mathcal{S} = (V, C')$ of $\mathcal{N}$ such that $C'(v, v') = \{b\}$.

Definition 3 *A* QCN $\mathcal{N} = (V, C)$ *is* minimal *iff* $\forall v, v' \in V$ *and* $\forall b \in C(v, v')$*, b is a feasible base relation of* $\mathcal{N}$*. The* minimal labeling problem (MLP) *is the problem of determining all the feasible base relations for each of the constraints of* $\mathcal{N}$*.*

Given a QCN $\mathcal{N} = (V, C)$, we say that a relation $C(v, v')$, with $v, v' \in V$, is non-redundant in $\mathcal{N}$, if there exists a base relation $b \notin C(v, v')$ and a solution σ of the QCN $\mathcal{N}' = (V, C')$ defined by $C'(v, v') = \mathsf{B} \setminus C(v, v')$, $C'(v', v) = \mathsf{B} \setminus (C(v, v'))^{-1}$, and $C'(u, w) = C(u, w)$ $\forall (u, w) \in (V \times V) \setminus \{(v, v'), (v', v)\}$ such that $(\sigma(v), \sigma(v')) \in b$. The relation is called non-redundant, because if we were to remove it and effectively replace it with relation B, the solution set of $\mathcal{N}$ would be changed. Note that by definition every universal relation B in a QCN is redundant.

Definition 4 *A* QCN $\mathcal{N} = (V, C)$ *is* reducible *iff it comprises a redundant relation other than relation* B*, and* irreducible *otherwise. The* redundancy problem *is the problem of obtaining all the non-redundant relations in* $\mathcal{N}$ *and, hence, determining if* $\mathcal{N}$ *is reducible.*

Checking the satisfiability of a QCN is $\mathcal{NP}$-complete in the general case for the most well-known and interesting calculi, such as RCC-8 (Renz and Nebel 1999) and IA (Nebel and Bürckert 1995). As a direct consequence, checking if a base relation of a QCN is feasible, or if a constraint of a QCN is non-redundant, is also $\mathcal{NP}$-complete in the general case. However, there exist *maximal tractable subclasses* $\mathcal{A} \subseteq 2^{\mathsf{B}}$ of the considered calculi, for which the satisfiability problem becomes tractable through the use of a *path-consistency*[4] algorithm.

With respect to subclasses of relations we have the following definition:

Definition 5 *A* subclass *of relations is a subset* $\mathcal{A} \subseteq 2^{\mathsf{B}}$ *that contains the singleton relations of* 2^{B} *and is closed under converse, intersection, and weak composition. A subclass* $\mathcal{A} \subseteq 2^{\mathsf{B}}$ *is a*

[3]This term is also found by the name "*refined* QCN" throughout the literature.

[4]The literature suggests the term *algebraic closure* (Renz and Ligozat 2005) instead, which is equivalent to a path-consistency algorithm where the weak composition operator $\diamond$ is used instead of the relational composition operator $\circ$ (Renz and Ligozat 2005), so we will use this more traditional term throughout the paper.

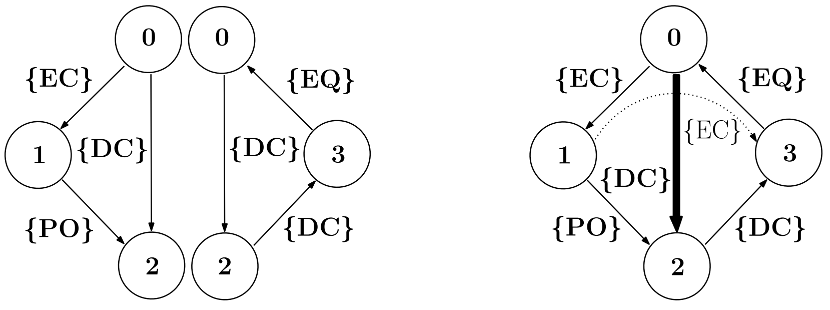


Figure 3 Patching two QCNs of RCC-8 that agree on their common part

tractable subclass *iff a* QCN *comprising relations solely from $\mathcal{A}$ is tractable. A tractable subclass $\mathcal{A} \subseteq 2^{\mathsf{B}}$ is* maximal *iff there exists no other tractable subclass that properly contains $\mathcal{A}$.*

Further, the notion of path-consistency for QCNs is defined as follows:

Definition 6 *A* QCN $\mathcal{N} = (V, C)$ *is* path-consistent *iff* $\forall v, v', v'' \in V$ *we have that* $C(v, v') \subseteq C(v, v'') \diamond C(v'', v')$.

Given a QCN $\mathcal{N} = (V, C)$, path-consistency can be applied on $\mathcal{N}$ in $O(|V|^3)$ time (Vilain, Kautz, and Beek 1990). For the considered calculi in this paper, we have that not trivially inconsistent and path-consistent QCNs defined on a maximal tractable subclass $\mathcal{A} \subseteq 2^{\mathsf{B}}$ are satisfiable (Nebel and Bürckert 1995; Renz and Nebel 1999).[5] The maximal tractable subclasses of relations of RCC-8 and IA are the classes $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$ (Renz and Nebel 2001) and $\mathcal{H}_{\mathsf{IA}}$ (Nebel 1997) respectively. Classes $\hat{\mathcal{H}}_8$ and $\mathcal{H}_{\mathsf{IA}}$ contain exactly those relations that are transformed to propositional <u>H</u>orn formulas when using the propositional encodings of RCC-8 and IA respectively. Further, and for RCC-8 in particular, if we denote by $\mathcal{P}_8$ the set of relations that belong to either one of the classes $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$, then all relations of $\mathcal{P}_8$ not contained in $\mathcal{C}_8$ contain $E\underline{C}$ and all relations of $\mathcal{P}_8$ not contained in $\mathcal{Q}_8$ contain $E\underline{Q}$ (Renz 1999). The propositional encoding of either $\mathcal{C}_8$ or $\mathcal{Q}_8$ is neither a Horn formula nor a Krom formula, but classes $\mathcal{C}_8$ and $\mathcal{Q}_8$ themselves are directly related to class $\hat{\mathcal{H}}_8$ in the sense that any QCN defined over either $\mathcal{C}_8$ or $\mathcal{Q}_8$ can be polynomially refined to a QCN defined over $\hat{\mathcal{H}}_8$ (Renz 1999).

Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$, we have that $\mathcal{N} \cup \mathcal{N}'$ yields the QCN $\mathcal{N}'' = (V'', C'')$, where $V'' = V \cup V'$, $C''(u, v) = C''(v, u) = \mathsf{B}$ for all $(u, v) \in (V \setminus V') \times (V' \setminus V)$, $C''(u, v) = C(u, v) \cap C'(u, v)$ for every $u, v \in V \cap V'$, $C''(u, v) = C(u, v)$ for all $(u, v) \in (V \times V) \setminus (V' \times V')$, and $C''(u, v) = C'(u, v)$ for all $(u, v) \in (V' \times V') \setminus (V \times V)$.

We now recall the definition of the *patchwork* property that was originally introduced in (Lutz and Milicic 2007) and was shown to be satisfied by IA and RCC-8 for satisfiable atomic QCNs of their relations.

Definition 7 *A constraint language has* patchwork, *iff for any finite satisfiable constraint networks* $\mathcal{N} = (V, C)$ *and* $\mathcal{N}' = (V', C')$ *defined on this language where* $\forall u, v \in V \cap V'$ *we have that* $C(u, v) = C'(u, v)$, *the constraint network* $\mathcal{N} \cup \mathcal{N}'$ *is satisfiable.*

Huang showed that IA and RCC-8 have patchwork for certain satisfiable non-atomic QCNs of their relations as well (Huang 2012). In particular, we have the following proposition:

Proposition 1 (Huang 2012) *The qualitative constraint languages of* IA *and* RCC-*8 have patchwork for not trivially inconsistent and path-consistent* QCN*s comprising relations solely from one of the maximal tractable subclasses* $\mathcal{H}_{\mathsf{IA}}$ *and* $\hat{\mathcal{H}}_8, \mathcal{C}_8$, *or* $\mathcal{Q}_8$ *respectively.*

[5]Some of the cited works are based on encodings of QCNs into Boolean formulas. However, the Boolean formulas are constructed in such a way that each solution of a formula corresponds to a not trivially inconsistent and path-consistent QCN with relations solely from some maximal tractable subclass of relations, and vice versa.

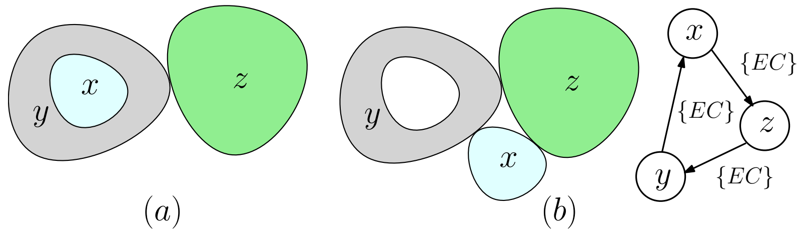


Figure 4 RCC-8 configurations

Intuitively, patchwork ensures that the combination of two satisfiable constraint networks that agree on their common part, i.e., on the constraints between their common variables, continues to be satisfiable. As an example, we can view the two QCNs of RCC-8 in Figure 3. (Self-loops corresponding to singleton relation $\{EQ\}$ and converses of constraints are not shown for simplicity.) The QCNs of RCC-8 are atomic, as they comprise singleton relations, and are also path-consistent, therefore, by Proposition 1 and application of the patchwork property their union is satisfiable, since they agree on the constraints between their common variables, namely, on $C_{02} = \{DC\}$. (The universal relation that exists by definition between variables 1 and 3 in the unified QCN would result to relation $C_{13} = \{EC\}$ if we were to calculate it, but it is not necessary to do so unless required by the specifics of the use case at hand.)

Patchwork is closely related to the *global consistency* property (Renz and Ligozat 2005), which is defined as follows:

Definition 8 *A* QCN $\mathcal{N} = (V, C)$ *is* globally consistent *if and only if, for any $V' \subset V$, every partial solution on V' can be extended to a partial solution on $V' \cup \{v\} \subseteq V$, for any $v \in V \setminus V'$.*

In particular, global consistency implies patchwork, but the opposite is not true. For example, even though RCC-8 has patchwork (Huang 2012), it does not have global consistency (Renz and Ligozat 2005). For instance, let us consider the spatial configuration shown in Figure 4(a). Region y is a doughnut, and region x is externally connected to it, by occupying its hole. Further, region z is externally connected to region y. With respect to RCC-8, we know that the constraint network defined by the set of constraints $\{EC(x, y), EC(y, z), EC(x, z)\}$ is satisfiable, as it is path-consistent and atomic. However, the valuation of region variables x and y is such that it is impossible to extend it with a valuation of region variable z so that $EC(x, z)$ may hold. Patchwork allows us to disregard any partial valuations and focus on the satisfiability of the network. Then, we can consider a valuation that satisfies the constraint network. Such a valuation is, for example, the one presented in Figure 4(b) along with its corresponding scenario.

3 Tree Decomposition and Chordal Graph

In this section, we recall the notions of a tree decomposition and a chordal graph, and review their use and effect in qualitative spatial and temporal reasoning in combination with patchwork[6].

In what follows, we consult the book of Diestel on Graph Theory for related definitions and properties (Diestel 2012). A tree decomposition is formally defined as follows:

Definition 9 *A* tree decomposition *of a graph $G = (V, E)$ is a tuple (T, X) where $T = (I, F)$ is a tree and $X = \{X_i \subseteq V \mid i \in I\}$ a collection of clusters (subsets of V) that satisfy the following properties:*

- *For every $v \in V$ there is at least one node $i \in I$ such that $v \in X_i$.*
- *For every $(u, v) \in E$ there exists a node $i \in I$ such that both $u, v \in X_i$.*
- *Let i_1, i_2, i_3 be three nodes in I such that i_2 lies on the path between i_1 and i_3 in T. Then, if $v \in V$ belongs to both X_{i_1} and X_{i_3}, v must also belong to X_{i_2}.*

[6]Some of the cited works use a property called *amalgamation*, which is equivalent to patchwork for satisfiable atomic networks when the satisfiability of atomic networks can be decided by path-consistency.

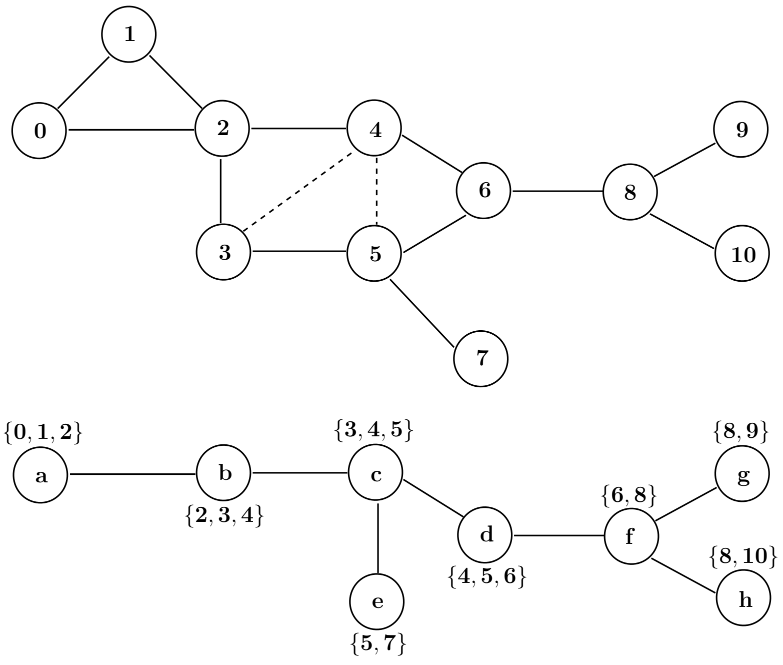


Figure 5 A graph (upper part) and its tree decomposition (lower part)

Let us view the example presented in Figure 5. In the upper part of the figure we can view a graph $G = (V, E)$, which can correspond to the structure of a constraint graph of a QCN. For the moment, we consider only the solid edges to be part of G and we disregard the dashed edges $(3, 4)$ and $(4, 5)$. A tree decomposition of G comprises a tree $T = (I, F)$ and a cluster X_i for every node $i \in I$ of that tree as shown in the lower part of the figure, e.g., $X_a = \{0, 1, 2\}$.

Tree decompositions have been explicitly introduced in qualitative reasoning by Condotta et al. in (Condotta and D'Almeida 2011), and implicitly by Li et al. in (Li, Huang, and Renz 2009) and Huang et al. in (Huang, Li, and Renz 2013). (The work presented in (Huang, Li, and Renz 2013) properly contains the work presented in (Li, Huang, and Renz 2009), thus, we will stick to the former reference in what follows.)

In (Condotta and D'Almeida 2011) the authors apply path-consistency on the clusters of a tree decomposition of the constraint graph of a QCN. The graphs induced by the clusters of the tree decomposition are completed with the introduction of a new set of edges, called *fill edges*, that correspond to the universal relation for a QCN. These fill edges for the example graph of Figure 5 are edges $(3, 4)$ and $(4, 5)$. As such, the clusters of the tree decomposition are considered to be *cliques*, namely, sets of vertices such that every two vertices in a set are connected by an edge. This is done for two reasons: (*i*) by definition path-consistency considers all triples of variables of a given constraint network and, hence, involves a complete graph, and (*ii*) the common vertices between any two complete graphs induce a complete graph, thus, the corresponding constraint networks will completely agree on the constraints between their common variables and the patchwork property can be used. Patchwork is then applied to patch together the not trivially inconsistent and path-consistent *pre-convex* QCNs of IA (Ligozat 2011) that correspond to the graphs induced by the clusters of the tree decomposition in a tree-like manner and construct a satisfiable network.

In (Huang, Li, and Renz 2013) the authors enlist a structure known as a *dtree* (decomposition tree), which, as the name suggests, is very close to a tree decomposition. Without going further into detail, a dtree is a full binary tree where the root represents a given graph and for each non-leaf node its two children represent a partitioning of the parent graph into two subgraphs. Thus, although a dtree is not a tree decomposition, it provides a way to construct a tree decomposition out of a given graph. A dtree and a tree decomposition are therefore equivalent in the context of qualitative reasoning, since omitting path-consistency checks across children of dtree nodes (as described in (Huang, Li, and Renz 2013)) corresponds to omitting those checks across clusters of the tree decomposition into which the dtree is converted, as has been specifically pointed out in (Condotta and D'Almeida 2011). Similarly to what is done in (Condotta and D'Almeida 2011), children of dtree nodes are treated as cliques, and patchwork is considered to patch together the path-consistent atomic QCNs of either IA or RCC-8 in a tree-like recursive manner and construct a satisfiable network.

The observant reader will note that it would be convenient to operate directly on a tree decomposition (T, X) of a given graph G where X would be a collection of cliques. In this context *chordal graphs* become relevant. Formally, a chordal graph is defined as follows:

Definition 10 *A graph G is said to be* chordal *(or triangulated) if every cycle of length greater than* 3 *has a chord, which is an edge connecting two non-adjacent nodes of the cycle.*

We then have the following proposition:

Proposition 2 (Diestel 2012) *A graph G is chordal if and only if it has a tree decomposition* $(T, \{X_1, \ldots, X_n\})$ *where cluster* X_i *is a clique of G for every* $i \in \{1, \ldots, n\}$.

For example, the graph presented in Figure 5, with the dashed edges included, is chordal. Chordality checking can be done in (linear) $O(|V| + |E|)$ time for a given graph $G = (V, E)$ with the *maximum cardinality search* algorithm, which also constructs an *elimination ordering* ω as a byproduct (Tarjan and Yannakakis 1984). If a graph is not chordal, it can be made so through the addition of fill edges. This process is usually called *triangulation* of a given graph $G = (V, E)$ and can run as fast as in $O(|V| + (|E \bigcup F(\omega)|))$ time, where $F(\omega)$ is the set of fill edges that results by following the elimination ordering ω, eliminating the nodes one by one, and connecting all nodes in the neighborhood of each eliminated node, thus, making it simplicial in the resulting subgraph. If the graph is already chordal, following the elimination ordering ω produced by the maximum cardinality search algorithm guarantees that no fill edges are added, i.e., ω is actually a *perfect elimination ordering* (Diestel 2012). For example, a perfect elimination ordering for the chordal graph shown in Figure 5 would be the ordering $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 7 \rightarrow 5 \rightarrow 6 \rightarrow 9 \rightarrow 8 \rightarrow 10$ of its set of nodes. In general, it is desirable to achieve chordality with as few fill edges as possible. However, triangulating a graph with the minimum number of fill edges is known to be $\mathcal{NP}$-complete (Yannakakis 1981). As noted earlier, fill edges correspond to the universal relation for a QCN. As such, the *chordal constraint graph* of a given QCN is exactly its constraint graph augmented with constraints corresponding to the universal relation to make it chordal.

In light of Propositions 1 and 2, research efforts focused on making the constraint graph of a given QCN chordal and restricting path-consistency to that chordal graph, while fully utilizing maximal tractable subclasses of relations and not just base relations that are typically used to describe only atomic networks. Towards this direction, we have the works of Chmeiss et al. for IA (Chmeiss and Condotta 2011) and Sioutis et al. for RCC-8 (Sioutis and Koubarakis 2012). These works were later combined in (Amaneddine, Condotta, and Sioutis 2013) to give the following result, which is the strongest yet concerning path-consistency, patchwork, and maximal tractable subclasses of relations:

Proposition 3 (Amaneddine, Condotta, and Sioutis 2013) *For a given* QCN $\mathcal{N} = (V, C)$ *defined over a tractable subclass of relations of a qualitative constraint language that has patchwork for not trivially inconsistent and path-consistent* QCN*s comprising relations solely from a tractable subclass of its relations, and for* $G = (V, E)$ *a triangulation of its constraint graph, if* $\forall (i, j), (i, k), (j, k) \in E$ *we have that* $C_{ij} \subseteq C_{ik} \diamond C_{kj}$*, then* $\mathcal{N}$ *is satisfiable.*

Consequently, by Propositions 1 and 3 we have the following result:

Corollary 1 *For a given* QCN $\mathcal{N} = (V, C)$ *of* RCC-*8, or* IA*, with relations solely from one of the maximal tractable subclasses* $\hat{\mathcal{H}}_8, \mathcal{C}_8$*, and* $\mathcal{Q}_8$*, or* $\mathcal{H}_{\mathsf{IA}}$*, respectively, and for* $G = (V, E)$ *a*

Algorithm 1: PartialPathConsistency($\mathcal{N}$, G)

```
in     : A QCN $\mathcal{N} = (V, C)$, and a graph $G = (V, E)$.
output : False if the constraint network $\mathcal{N}$ results in a trivial inconsistency (contains the empty relation),
         True if the (possibly) refined constraint network $\mathcal{N}$ is partially path consistent.
begin
    if $\exists(i, j) \in E(\mathsf{G}(\mathcal{N}))$ with $C_{ij} = \emptyset$ then
        return False;
    $Q \leftarrow \{(i, j) \mid (i, j) \in E\}$;
    while $Q \neq \emptyset$ do
        $(i, j) \leftarrow Q.pop()$;
        foreach $k$ such that $(i, k), (k, j) \in E$ do
            $t \leftarrow C_{ik} \cap (C_{ij} \diamond C_{jk})$;
            if $t \neq C_{ik}$ then
                if $t = \emptyset$ then
                    return False;
                $C_{ik} \leftarrow t$; $C_{ki} \leftarrow t^{-1}$;
                $Q \leftarrow Q \cup \{(i, k)\}$;
            $t \leftarrow C_{kj} \cap (C_{ki} \diamond C_{ij})$;
            if $t \neq C_{kj}$ then
                if $t = \emptyset$ then
                    return False;
                $C_{kj} \leftarrow t$; $C_{jk} \leftarrow t^{-1}$;
                $Q \leftarrow Q \cup \{(k, j)\}$;
    return True;
```

triangulation of its constraint graph, if $\forall(i, j), (i, k), (j, k) \in E$ *we have that* $C_{ij} \subseteq C_{ik} \diamond C_{kj}$*, then* $\mathcal{N}$ *is satisfiable.*

Proposition 3 generalizes the results of all the works that were discussed earlier in this section and make use of path-consistency as the main tool for checking the satisfiability of a given QCN, and has a great effect in the efficiency and scalability of practical reasoning. In particular, regarding native search, an algorithm based on the work of (Bliek and Sam-Haroud 1999) was devised, called *partial path-consistency* (Chmeiss and Condotta 2011), that enforces *partial path-consistency* on a given QCN $\mathcal{N} = (V, C)$ with respect to a triangulation $G = (V, E)$ of its constraint graph in $O(\delta|E|)$ time, where δ is the maximum vertex degree of G.

Definition 11 *Given a* QCN $\mathcal{N} = (V, C)$ *and a graph* $G = (V, E)$*,* $\mathcal{N}$ *is* partially path-consistent *(with respect to graph G) iff for* $\forall(v, v'), (v, v''), (v'', v') \in E$ *we have that* $C(v, v') \subseteq C(v, v'') \diamond C(v'', v')$*.*

The partial path-consistency algorithm of Chmeiss and Condotta is presented in Algorithm 1. As it is suggested by Proposition 3, the partial path-consistency algorithm is able to decide the satisfiability of a QCN when it comprises relations solely from some maximal tractable subclass of relations, when path-consistency can yield patchwork with respect to that subclass of relations, and when a triangulation of its constraint graph is used as the input graph in the algorithm.

The search space for non-tractable QCNs was also reduced to $O(\alpha^{|E|})$ from $O(\alpha^{|V|^2})$ for a backtracking algorithm (Renz and Nebel 2001), where α is the branching factor provided by some maximal tractable subclass of relations (e.g., $\alpha = 1.4375$ for class $\hat{\mathcal{H}}_8$ of RCC-8 (Renz and Nebel 2001)). Such a backtracking algorithm is presented in Algorithm 2. Note that if a QCN $\mathcal{N}$ along with a triangulation G of its constraint graph and a maximal tractable subclass $\mathcal{A}$ are given as input to the backtracking algorithm, then the algorithm is able to decide the satisfiability of the given network $\mathcal{N}$ provided that path-consistency can yield patchwork with respect to $\mathcal{A}$.

Regarding approaches based on encodings of QCNs into Boolean formulas, i.e., SAT-based approaches, the implication of Proposition 3 led to significant memory and speed improvements both for IA (Westphal, Hué, and Wölfl 2013) and for RCC-8 (Westphal and Hué 2014) targeted implementations. Further, regarding works that consider the MLP and the redundancy problem of a QCN, partial path-consistency has been used as the core local consistency condition to build

Algorithm 2: PartialConsistency($\mathcal{N}$, G, $\mathcal{A}$)

```
in     : A QCN $\mathcal{N} = (V, C)$, a graph $G = (V, E)$, and a subclass $\mathcal{A}$.
output : Null if the constraint network $\mathcal{N}$ results in a trivial inconsistency, a (possibly) refined constraint
         network $\mathcal{N}$ over $\mathcal{A}$ otherwise.
1  begin
2      if ! PartialPathConsistency($\mathcal{N}$, $G$) then
3          return Null;
4      if $\forall(i, j) \in E$ we have that $C_{ij} \in \mathcal{A}$ then
5          return $\mathcal{N}$;
6      choose constraint $C_{ij}$ such that $C_{ij} \notin \mathcal{A}$;
7      split $C_{ij}$ into $r_1, ..., r_k \in \mathcal{A}$: $r_1 \cup ... \cup r_k = C_{ij}$;
8      foreach $r \in \{r_l \mid 1 \leq l \leq k\}$ do
9          replace $C_{ij}$ with $r$ in $\mathcal{N}$;
10         result ← PartialConsistency($\mathcal{N}$, $G$, $\mathcal{A}$);
11         if result $\neq$ Null then
12             return result;
13     return Null;
```

algorithms both for the MLP as described in (Amaneddine, Condotta, and Sioutis 2013) and for the redundancy problem as described in (Sioutis, Li, and Condotta 2015).

Before closing this section with some strong theoretical results that concern tree decompositions and patchwork, let us introduce the *treewidth* of a graph. The *width* of a tree decomposition $(T, \{X_1, \ldots, X_n\})$ is $\max_{1 \leq i \leq n} |X_i| - 1$. The *treewidth* of a graph G is the minimum width possible for arbitrary tree decompositions of G. We also recall the following result regarding the treewidth of a graph G that is augmented with a new edge:

Theorem 1 (Elidan and Gould 2008) *Let $G = (V, E)$ be a graph of treewidth k. Then, the treewidth of graph $G' = (V, E \cup \{e\})$, where e is a new edge, is at most $k + 1$.*

In the context of QCNs, the treewidth of a QCN $\mathcal{N}$ is simply the treewidth of its constraint graph $\mathsf{G}(\mathcal{N})$.

Theorem 2 (Bodirsky and Wölfl 2011; Huang, Li, and Renz 2013) *For any k, the satisfiability problem for* QCN*s of treewidth at most k that are defined on a language that has patchwork for path-consistent atomic* QCN*s can be solved in polynomial time.*

Consequently, by Proposition 1 and Theorem 2 we have the following result:

Corollary 2 *For any k, the satisfiability problem for* QCN*s of* IA *and* RCC-*8 of treewidth at most k can be solved in polynomial time.*

A detailed algorithm for Theorem 2 that offers an alternative to the proof sketch of Bodirsky et al. in (Bodirsky and Wölfl 2011) is provided in (Huang, Li, and Renz 2013). (The proof sketch in (Bodirsky and Wölfl 2011) is particular to RCC-8, but it can be generalized to IA and other languages satisfying certain common properties.)

Further, regarding the MLP we can have the following result:

Theorem 3 *For any k, the* MLP *for* QCN*s of treewidth at most k that are defined on a language that has patchwork for path-consistent atomic* QCN*s can be solved in polynomial time.*

Proof Let $\mathcal{N} = (V, C)$ be a QCN of treewidth at most k that is defined on a language that has patchwork for path-consistent atomic QCNs. To check if a base relation $b \in C(u, v)$, with $(u, v) \in \mathsf{G}(\mathcal{N})$, participates in at least one solution of $\mathcal{N}$, we must check if the QCN $\mathcal{N}' = (V, C')$ defined by $C'(u, v) = \{b\}$, $C'(v, u) = \{b\}^{-1}$, and $C'(y, w) = C(y, w)$ $\forall(y, w) \in (V \times V) \setminus \{(u, v), (v, u)\}$ is satisfiable, so that a scenario $S = (V, C'')$ with $C''(u, v) = \{b\}$ can be constructed out of the admitted solution. This satisfiability check can be done in polynomial time by Theorem 2. Note that if $(u, v) \notin \mathsf{G}(\mathcal{N})$, we must augment the constraint graph $\mathsf{G}(\mathcal{N})$ with (u, v) to take into account the constraint $C(u, v)$. As such, the satisfiability check will be performed on a QCN of treewidth at most $k + 1$ due to Theorem 1. (After the check, (u, v) can again be removed from $\mathsf{G}(\mathcal{N})$.) As we can have at most $O(|\mathsf{B}||V|^2)$ base relations in any given QCN, it follows that we can solve the MLP in polynomial time. □

Consequently, by Proposition 1 and Theorem 3 we have the following result:

Corollary 3 *For any k, the* MLP *for* QCN*s of* IA *and* RCC-*8 of treewidth at most k can be solved in polynomial time.*

Regarding the redundancy problem, we can have the following result:

Theorem 4 *For any k, the redundancy problem for* QCN*s of treewidth at most k that are defined on a language that has patchwork for path-consistent atomic* QCN*s can be solved in polynomial time.*

Proof Let $\mathcal{N} = (V, C)$ be a QCN of treewidth at most k that is defined on a language that has patchwork for path-consistent atomic QCNs. To check if a constraint $C(u, v)$, with $(u, v) \in \mathsf{G}(\mathcal{N})$, is non-redundant in $\mathcal{N}$, we must check if there exists a base relation $b \notin C(u, v)$ that participates in a solution of the modified $\mathcal{N}$ that results by removing $C(u, v)$. This is equivalent to checking if the QCN $\mathcal{N}' = (V, C')$ defined by $C'(u, v) = \mathsf{B} \setminus C(u, v)$, $C'(v, u) = \mathsf{B} \setminus (C(u, v))^{-1}$, and $C'(y, w) = C(y, w)\ \forall (y, w) \in (V \times V) \setminus \{(u, v), (v, u)\}$ is satisfiable. This satisfiability check can be done in polynomial time by Theorem 2. (Note that if $(u, v) \notin \mathsf{G}(\mathcal{N})$, $C(u, v)$ is by definition redundant.) Since we can have at most $O(|V|^2)$ constraints in any given QCN, it follows that we can solve the redundancy problem in polynomial time. □

Consequently, by Proposition 1 and Theorem 4 we have the following result:

Corollary 4 *For any k, the redundancy problem for* QCN*s of* IA *and* RCC-*8 of treewidth at most k can be solved in polynomial time.*

4 Partitioning Graph

In this section, we prove that the decomposition-based approach presented in (Nikolaou and Koubarakis 2014) for checking the satisfiability of QCNs of RCC-8 lacks soundness, as the notion of a *partitioning graph* defined in that work is not coherent with the use of patchwork upon which it solely relies, in two ways, which we enumerate and analyse in the form of issues.

Let $G = (V, E)$ be a graph and k a positive integer. If $U \subseteq V$, then $G(U)$ will denote the subgraph of G that is induced by the set of vertices U. A set $\{V_i \subseteq V \mid 1 \leq i \leq k\}$ with k pairwise-disjoint elements such that $\bigcup_{i=1}^{k} V_i = V$, is called a k-way partitioning of G. Finally, let $\emptyset$ denote the empty, edgeless, graph. We recall the following definition of a partitioning graph from (Nikolaou and Koubarakis 2014):

Definition 12 *Let $G = (V, E)$ be a graph and $\{V_1, \ldots, V_k\}$ a k-way partitioning of G for some positive integer k. A* partitioning graph *P of G is a graph $(V_P, E_P, \lambda_P, G_P)$, where $V_P = \{v_1, \ldots, v_k\}$ is the set of its nodes, E_p the set of its edges, $\lambda_P : V_P \rightarrow 2^V$ a function that maps each node of P to a partition (subset of V) of G, and G_P a set of k subgraphs (parts) of G. The following conditions must be satisfied:*

- *If $G_i \in G_P$ then the set of vertices of G_i is a superset U of $\lambda_P(v_i)$ and the set of its edges is $E(G(U))$.*
- *Any edge in G should be present in at least one subgraph $G_i \in G_P$.*
- *An edge (v_i, v_j) belongs to E_P if and only if $G_i \cap G_j \neq \emptyset$ (i.e., if and only if the subgraphs G_i and G_j corresponding to nodes v_i and v_j respectively share a common edge).*

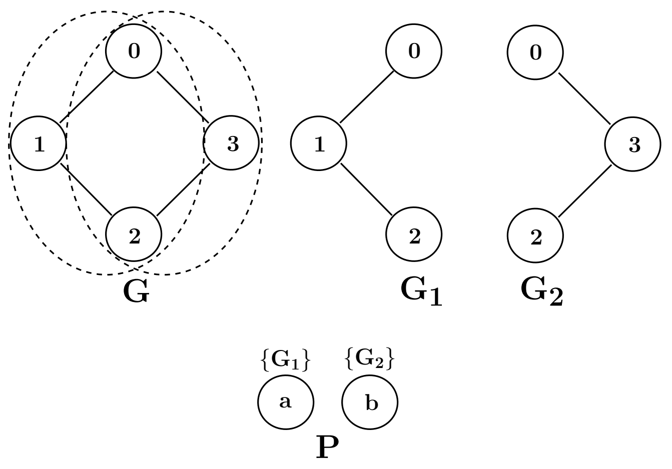


Figure 6 A graph and its partitioning graph with the parts comprising it (also contained in dashed circles in the initial graph)

Let G be a graph and $P = (V_P, E_P, \lambda_P, G_P)$ its partitioning graph. Then, an edge e of G present in more than one subgraph $G_i \in G_P$ is called a *global edge*. An edge e of G present in exactly one subgraph $G_i \in G_P$ is called a *local edge*.

We will now enumerate the issues that lead to non-soundness and provide counter-examples for each case. The reader is kindly asked to refer to (Nikolaou and Koubarakis 2014) and check that the flaws pointed out here are actually present in (Nikolaou and Koubarakis 2014).

The issues that we will enumerate will allow us to infer the following fact:

Proposition 4 *The approach presented in (Nikolaou and Koubarakis 2014) for checking the satisfiability of a* QCN *of* RCC-*8 lacks soundness. In particular, Propositions* 2 *and* 3 *in (Nikolaou and Koubarakis 2014) do not hold.*

We begin with the first issue.

Issue 1. The first issue has to do with the fact that a complete agreement on the constraints between the common variables of two networks is not achieved in order to allow the applicability of patchwork. Let us consider the example of Figure 6. Graph G is partitioned into two parts, namely, G_1 and G_2. The partitioning graph is shown in the lower part of the figure, and it comprises the set of nodes $\{a, b\}$ and an empty set of edges. Node a corresponds to subgraph G_1 and node b to subgraph G_2. Its set of edges E_P is empty as subgraphs G_1 and G_2 do not share a common edge (that would otherwise be the global edge $(0, 2)$), thus, the only possible edge (a, b) does not exist. In (Nikolaou and Koubarakis 2014) the authors perform path-consistency on the subgraphs of a graph separately, in a parallel fashion, and then rely on the set of edges E_P to identify the subgraphs among which a complete agreement has to be ensured (the reader is kindly asked to refer to line 7 in the function of Algorithm 2 in (Nikolaou and Koubarakis 2014)). If, as in this example, such an edge does not exist, a complete agreement is never achieved. This can be the cause of failing to identify inconsistencies. Let us assume that graph G, as depicted in Figure 6, is the constraint graph of a given QCN comprising constraints $C_{01} = C_{12} = C_{23} = C_{30} = \{TPP\}$. This yields an unsatisfiable network, as it basically infers that region 0 is properly contained in region 2, and vice versa. Applying path-consistency on that network would result in the empty relation assignment for constraint C_{02} (inconsistency). However, that constraint is never checked in our example. Although the authors implicitly complete subgraphs G_1 and G_2 in order to apply path-consistency, they do not complete these subgraphs when computing their intersection, as clearly specified in the last bullet of Definition 12. Even if they did implicitly consider complete subgraphs for that part of the definition, and the edge (a, b) indeed existed, line 7 in the function of Algorithm 2 in (Nikolaou and Koubarakis 2014) still requires that an agreement should only be achieved for every common edge of G_1 and G_2 (the initial non-complete subgraphs), which is

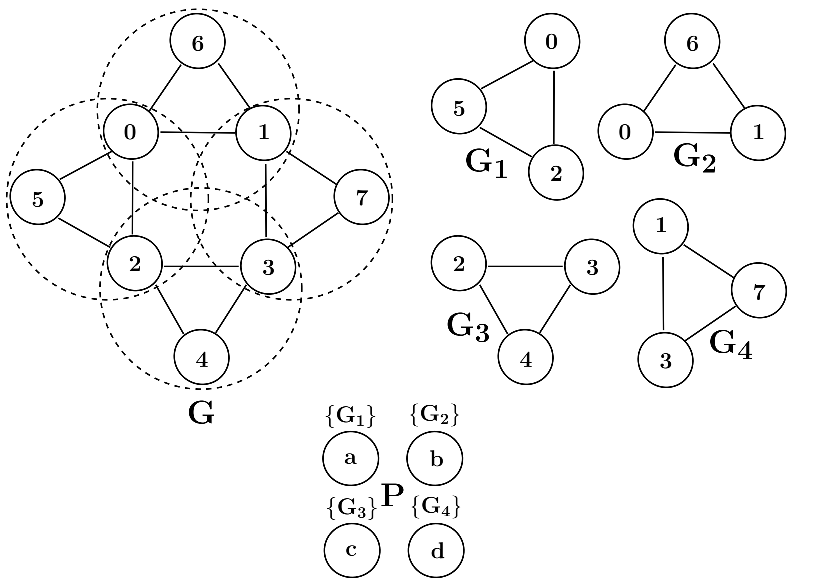


Figure 7 A graph and its partitioning graph with the parts comprising it (also contained in dashed circles in the initial graph)

none. If they implicitly considered complete subgraphs for that part of the algorithm too, then this particular issue for a 2-way partitioning would be resolved. We have also verified this issue experimentally with the implementation used in (Nikolaou and Koubarakis 2014).[7]

Before proceeding to the next issue, let us assume that the first issue is fixed with everything that we propose, and a 2-way partitioning is actually valid for applying patchwork. We mean to show, that the concept of a partitioning graph is beyond repair, unless it is structured in a way that it defines a tree decomposition, which defeats the purpose of having to define a partitioning graph in the first place.

The second issue follows.

Issue 2. This issue has to do with the fact that even if the first issue is resolved, the partitioning graph can suffer from the existence of cycles that are created by subgraphs of a given graph. Let us consider the example of Figure 7. Graph G is partitioned into four parts, namely, G_1, G_2, G_3, and G_4. The partitioning graph is shown in the lower part of the figure, and the correspondence between its sets of nodes and edges with the different subgraphs should be clear up to this point. Note that all subgraphs are complete, thus, they completely overlap with each other on the common vertices. For example, graph G_1 completely overlaps with graph G_2 on the edges of the graph defined on the single common vertex 0, as their intersection yields the complete graph on singe vertex 0. Although such an overlap is trivial, as a complete graph on a single vertex (singleton graph) does not have any edges, it is sufficient to ensure the applicability of patchwork for the corresponding constraint networks. (Our example can be easily extended to non-trivial overlaps.) However, due to the last bullet of Definition 12, the partitioning graph is unable to obtain any edges, as there can exist no global edges. In fact, even if some edges existed in E_P, in any possible combination and amount, the partitioning graph would still fail to detect the cycle that is constructed by the complete subgraphs G_1, G_2, G_3, and G_4, namely, the cycle defined by vertices 0, 1, 2, and 3. This cycle, as shown in the example of Figure 6, can harbor an inconsistency. Such a cycle exists also in (Nikolaou and Koubarakis 2014, Fig. 1) between vertices 3, 4, 5, and 7 there. Patchwork alone is only valid for tree decompositions, as tree decompositions guarantee acyclicity of cliques and, thus, do not harbor cycles with potential inconsistencies that cannot be detected by the application of path-consistency on the different cliques. This issue was again verified experimentally.

[7]`https://www.dropbox.com/sh/h61edhshw5p8ne2/AAAuO0WyYB5r8cLmoRBLV8xla?dl=0`

Essentially, the approach defines a partial algorithm; a given satisfiable QCN will be shown to be satisfiable, as the approach in (Nikolaou and Koubarakis 2014) due to disregarding constraints operates on a less restrictive constraint graph of the input network where constraint propagation and consistency checks are limited, whilst an unsatisfiable QCN may be shown to be satisfiable.

4.1 Impact on Performance

The main contribution of (Nikolaou and Koubarakis 2014) lies in the performance of its offered implementation, as it promises efficiency that goes well beyond the state-of-the-art. Computing a good k-way partitioning[8] alone is among the graph partitioning problems that fall under the category of $\mathcal{NP}$-hard problems (Garey, Johnson, and Stockmeyer 1976), and solutions to these problems are generally derived using heuristics and approximation algorithms, such as the ones offered by the METIS[9] software employed in (Nikolaou and Koubarakis 2014). We leave aside any extra computational complexity that would result from needing to restrict a partitioning graph to being a tree decomposition (e.g., by identifying cycles or using some recursion as in (Huang, Li, and Renz 2013)), and focus on native search. As explained in Section 3, native search in qualitative spatial and temporal reasoning is bound to the number of constraints of a given QCN, and not to its number of variables as in "traditional" constraint programming. This is because, in a sense, the constraints of a given QCN are the *true* variables for which we have to assign some relation. Indeed, the search space defined in (Nikolaou and Koubarakis 2014) relies mainly on the number of constraints of a given QCN. In particular, we can recall the following proposition from (Nikolaou and Koubarakis 2014):

Proposition 5 (Nikolaou and Koubarakis 2014) *Let $G = (V, E)$ be the constraint graph of a* QCN *of* RCC-*8 and P a partitioning graph of G with k parts. The search space of algorithm* DConsistency *(Nikolaou and Koubarakis 2014) is $O(|\mathsf{B}|^g(|\mathsf{B}|gkm^3 + k\alpha^l m^3))$, where g is the number of global edges, l and m the maximum number of local edges and vertices respectively among all parts of P, and α the branching factor of the maximal tractable subclass of relations employed. If Π denotes the aforementioned search space, then given p processing units and assuming a balanced partitioning among the k parts (i.e., $m = |V|/k$), the elapsed running time of algorithm* DConsistency *is $O(\frac{\Pi}{p})$.*

We showed earlier that some global edges can be disregarded, thus, parameter g as defined in Proposition 5 leads to a significantly reduced search space for the implementation of (Nikolaou and Koubarakis 2014) with respect to the one that should normally be considered, as g has an exponential contribution. However, even in that case, a re-evaluation of the implementation used in (Nikolaou and Koubarakis 2014) against state-of-the-art solvers, showed that it performs very poorly with respect to the state-of-the-art (Sioutis 2014). The work in (Sioutis 2014) does not deal with any of the issues that we dealt with in this paper as it assumes a partitioning graph to implicitly define a tree decomposition, thus, (Sioutis 2014) presents mostly lower bounds on the performance of the implementation used in (Nikolaou and Koubarakis 2014).

4.2 Fixing the Issues

We noted earlier that the concept of a partitioning graph is beyond repair, unless it is structured in a way that it defines a tree decomposition. It may seem tempting as a quick hack to triangulate the constraint graph of a given QCN of RCC-8 and, thus, obtain a chordal constraint graph of that QCN, and feed it directly to the partitioning algorithm described in (Nikolaou and Koubarakis 2014). This may still yield non-soundness; we explain as follows. Consider the example shown in Figure 8 where the chordal graph G is partitioned into 3 subgraphs. The partitioning graph P is

[8] *Good* in terms of obtaining smaller components that meet specific properties, for example, a good partitioning can be defined as one in which the number of edges running between separated components is small.

[9] `http://glaros.dtc.umn.edu/gkhome/views/metis`

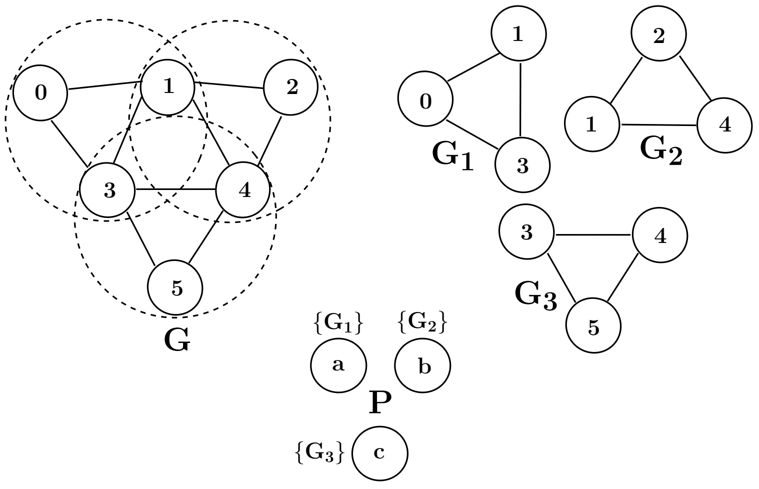


Figure 8 A chordal graph and its partitioning graph with the parts comprising it (also contained in dashed circles in the initial graph)

such that it is unable to capture/break the cycle defined by vertices 1, 3, and 4. This cycle may harbor an inconsistency, which will not be detected by the application of path-consistency on the different parts of the partitioning graph. One way to force a partitioning graph into defining a tree decomposition is using METIS in a recursive manner, as it is done in (Huang, Li, and Renz 2013). In particular, one has to initially partition a given graph G into two parts, and then recursively apply the same procedure on the obtained parts, until no further partitioning can occur. However, this can be a costly operation. A faster way is to rely on chordal graphs (tree decompositions into cliques), which can both be constructed and also yield a natural tree decomposition of their cliques in linear time (Diestel 2012). The graphs induced by the cliques can then be collected at no extra cost and serve as the parts of the partitioning graph; consequently, the approach described in (Nikolaou and Koubarakis 2014) can then be carried out with soundness and completeness.

5 Towards Efficient Utilization of Parallelism

As noted, in (Nikolaou and Koubarakis 2014) the authors provided a parallel implementation for checking the satisfiability of a QCN of RCC-8, which however lacks soundness (Proposition 4). In this section, we present a simple decomposition scheme that exploits the sparse and loosely connected structure of the constraint graphs of very large real world QCNs, which have been of high interest in the recent literature (Nikolaou and Koubarakis 2014; Sioutis and Condotta 2014; Sioutis 2014), and paves the way for efficient utilization of parallelism. Our approach is based on extracting the smaller QCNs that correspond to the biconnected components of the constraint graph of a given large QCN and reasoning with these smaller biconnected QCNs completely separately, in a parallel or serial fashion, which, as our experimentation suggests, significantly decongests search when solving non-tractable QCNs.

First, we recall a definition from (Dechter 2003) regarding biconnected graphs and components.

Definition 13 *A connected graph G is said to have an* articulation vertex *u if there exist vertices v and v' such that all paths connecting v and v' pass through u. A graph that has an articulation vertex is called* separable*, and one that has none is called* biconnected*. A maximal subgraph with no articulation vertices is called a* biconnected component*.*

Intuitively, an articulation vertex is any vertex whose removal increases the number of connected components in a given graph. From (Dechter 2003) we also have the following property:

Property 1 (Dechter 2003) *Let G be a graph and $\{G_1, \ldots, G_n\}$ its biconnected components. Then, there exists a* tree decomposition *$(T, \{X_1, \ldots, X_n\})$ of G, where cluster $X_i \subseteq V(G)$ induces the biconnected component G_i of G, for every $i \in \{1, \ldots, n\}$.*

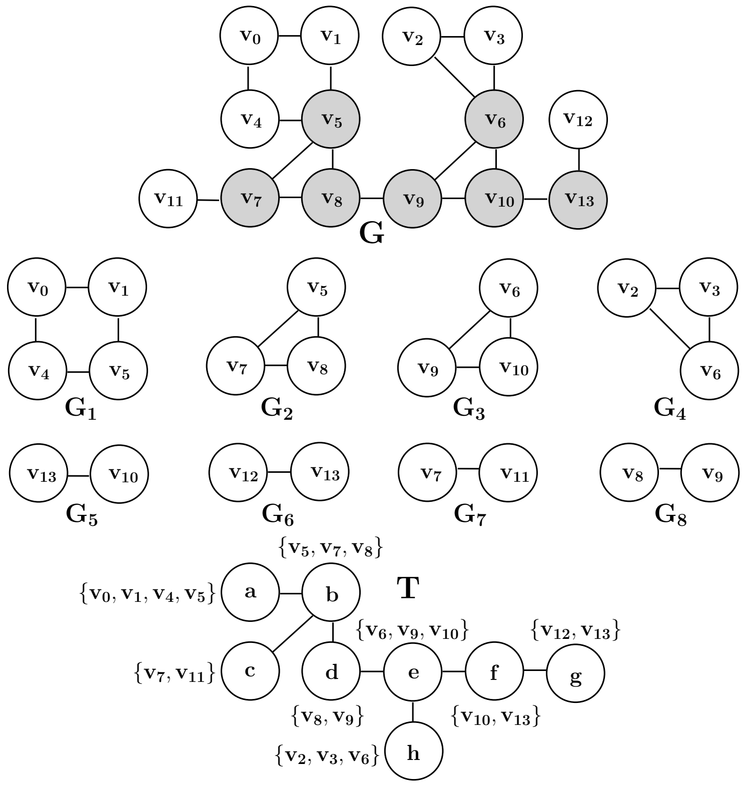


Figure 9 A graph G (top) with its biconnected components (middle) and its tree decomposition (bottom)

Let us now view the discussed notions in an example. Figure 9 depicts a graph G, along with its biconnected components, and its tree decomposition. Vertices in grey color are the articulation vertices of G. The tree decomposition comprises a tree $T = (I, F)$ and a cluster X_i for every node $i \in I$ of that tree, e.g., $X_a = \{v_0, v_1, v_4, v_5\}$. We can obtain the following proposition:

Proposition 6 *Let $\mathcal{N}$ be a* QCN *defined on a language that has patchwork for satisfiable atomic* QCN*s, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph* $\mathsf{G}(\mathcal{N})$*. Then, $\mathcal{N}$ is satisfiable iff $\mathcal{N}_i$ is satisfiable for every $i \in \{1, \ldots, k\}$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow_{V(G_i)}}$.*

Proof By Property 1, the constraint graph $\mathsf{G}(\mathcal{N})$ has a tree decomposition $(T, \{X_1, \ldots, X_k\})$, where cluster X_i induces G_i, for every $i \in \{1, \ldots, k\}$. We can also infer by Definition 13, that $\forall i, j \in \{1, \ldots, k\}$ with $i \neq j$, $V(G_i) \cap V(G_j)$ contains at most one vertex u (an articulation vertex). If $\mathcal{N}_i$ is satisfiable for every $i \in \{1, \ldots, k\}$, we can obtain a satisfiable atomic sub-QCN of $\mathcal{N}_i$, i.e., a scenario S_i of $\mathcal{N}_i$, for every $i \in \{1, \ldots, k\}$. For any possible scenarios, we will have that $\mathcal{S}_i = (V(G_i), C_i)$ and $\mathcal{S}_j = (V(G_j), C_j)$ will always agree on the single unary constraint that is defined by a single vertex $u \in V(G_i) \cap V(G_j)$ whenever we have that $V(G_i) \cap V(G_j) \neq \emptyset$, as by Definition 1 we have that $C_i(u, u) = C_j(u, u) = \{\mathsf{Id}\}$ $\forall u \in V(\mathsf{G}(\mathcal{N}))$, for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$. By Definition 7, we can apply patchwork to patch together all the satisfiable atomic QCNs $\mathcal{S}_i$ with $i \in \{1, \ldots, k\}$ in a tree-like manner and, thus, derive the satisfiability of $\mathcal{N}$. If $\mathcal{N}$ is satisfiable, then, clearly, $\mathcal{N}_i$ will be satisfiable for every $i \in \{1, \ldots, k\}$. □

Consequently, by Proposition 6 and the fact that IA and RCC-8 have patchwork for satisfiable atomic QCNs of their relations (Lutz and Milicic 2007), we have the following result:

Corollary 5 *Let $\mathcal{N}$ be a* QCN *of* IA *or* RCC-*8, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph* $\mathsf{G}(\mathcal{N})$*. Then, $\mathcal{N}$ is satisfiable iff $\mathcal{N}_i$ is satisfiable for every $i \in \{1, \ldots, k\}$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow_{V(G_i)}}$.*

Algorithm 3: Decomposer($\mathcal{N}$)

```
in      : A QCN $\mathcal{N} = (V, C)$.
output  : A collection of QCNs.
1  begin
2      if $\exists(v, v') \in E(\mathsf{G}(\mathcal{N}))$ with $C(v, v') = \emptyset$ then
3          return $\{(V, C)\}$;
4      $S \leftarrow \{g \mid g \in \mathsf{BCSubgraphs}(\mathsf{G}(\mathcal{N}))$; and $|V(g)| > 2\}$;
5      $\chi \leftarrow \emptyset$;
6      while $S \neq \emptyset$ do
7          $g \leftarrow S.\mathsf{pop}()$;
8          $V_g \leftarrow V(g)$; $E_g \leftarrow E(g)$;
9          $C_g \leftarrow \mathsf{map}(\{((v, v') : (\mathsf{B}$ if $v \neq v'$ else $\{\mathsf{Id}\})) \mid v, v' \in V_g\})$;
10         foreach $(v, v') \in E_g$ do
11             $C_g(v, v') \leftarrow C(v, v')$; $C_g(v', v) \leftarrow C(v', v)$;
12         $\chi \leftarrow \chi \cup \{(V_g, C_g)\}$;
13     return $\chi$;
```

Algorithm 4: Solver+($\mathcal{N}$)

```
in      : A QCN $\mathcal{N} = (V, C)$.
output  : True, or False.
1  begin
2      foreach $n \in$ Decomposer($\mathcal{N}$) do
3          if ! || Solver($n$) || then
4              return False;
5      return True;
```

It is important to note that the proof of Proposition 6 is based on tree decompositions whose nodes correspond to clusters where any two clusters share at most one vertex with each other. In case two clusters share more than one vertex with each other, the involved QCNs should be, for instance (and among other conditions), not trivially inconsistent, path-consistent, and defined over a tractable subclass of relations of a qualitative constraint language that has patchwork for not trivially inconsistent and path-consistent QCNs comprising relations solely from a tractable subclass of its relations, as it is specified in Proposition 3 and considered in (Sioutis and Koubarakis 2012) and (Chmeiss and Condotta 2011) for RCC-8 and IA respectively. A simple algorithm for obtaining a collection of QCNs that correspond to the biconnected components of the constraint graph of a given QCN is presented in Algorithm 3. Note that in lines 2–3 we immediately return the input QCN if it is trivially inconsistent, as it would not make any sense to continue with the decomposition procedure. Function $\mathsf{BCSubgraphs}(G)$ in line 4 returns the biconnected components of a graph $G = (V, E)$ and has a runtime of $O(|E|)$ (Dechter 2003). Note that in line 4 we keep only the components of order greater than 2, as any not trivially inconsistent qualitative constraint network of less than 3 variables is trivially satisfiable (by definition of a base relation). In what follows, we always consider components of order greater than 2. Based on algorithm Decomposer, we can obtain an algorithm to increase the performance of any given state-of-the-art solver that is sound and complete for checking the satisfiability of a given QCN $\mathcal{N}$ defined on a language that has patchwork for satisfiable atomic QCNs; that algorithm is presented in Algorithm 4. Let us denote any such given state-of-the-art solver by Solver. Then, Algorithm 4 will use Solver to decide the satisfiability of the QCNs that correspond to the biconnected components of the constraint graph of $\mathcal{N}$. The enclosure with symbol || for Solver denotes the fact that Solver can be used in a parallel or serial fashion.

Regarding the MLP, we can have the following result:

Proposition 7 *Let $\mathcal{N} = (V, C)$ be a satisfiable QCN defined on a language that has patchwork for satisfiable atomic QCNs, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph $\mathsf{G}(\mathcal{N})$. Then, a base relation $b \in C(u, v)$, with $u, v \in V(G_i)$, is feasible (resp. unfeasible) iff there exists (resp. there does not exist) a scenario $\mathcal{S}_i = (V_i, C'_i)$ of $\mathcal{N}_i = (V_i, C_i)$ such that $C'_i(u, v) = \{b\}$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow}_{V(G_i)}$, for some $i \in \{1, \ldots, k\}$.*

Proof Since $\mathcal{N}$ is satisfiable, by Proposition 6 we know that there exists a solution of $\mathcal{N}_i$ and, hence, a scenario of $\mathcal{N}_i$. In addition, as G_i is a biconnected component of $\mathsf{G}(\mathcal{N})$, any scenario $S_i = (V_i, C'_i)$ of $\mathcal{N}_i$ is the restriction of some scenario $S = (V, C')$ of $\mathcal{N}$ to V_i, and any scenario $S = (V, C')$ of $\mathcal{N}$ is the extension of some scenario $S_i = (V_i, C'_i)$ of $\mathcal{N}_i$ to V. As such, the feasibility of b can be characterized by considering $\mathcal{N}_i$ instead of $\mathcal{N}$. □

Given a satisfiable QCN $\mathcal{N} = (V, C)$, Proposition 7 allows one to quickly characterize the feasibility of a base relation $b \in C(u, v)$, with $u, v \in V(G')$, where G' is a biconnected component of the constraint graph $\mathsf{G}(\mathcal{N})$. If $u, v \notin V(G')$ for any biconnected component G' of $\mathsf{G}(\mathcal{N})$, then b belongs to a constraint that is labeled with the universal relation B and its feasibility can still be efficiently characterized under certain conditions by a function similar to extractFeasible as described in (Amaneddine, Condotta, and Sioutis 2013).

Consequently, by Proposition 7 and the fact that IA and RCC-8 have patchwork for satisfiable atomic QCNs of their relations (Lutz and Milicic 2007), we have the following result:

Corollary 6 *Let $\mathcal{N} = (V, C)$ be a satisfiable* QCN *of* IA *or* RCC-*8, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph $\mathsf{G}(\mathcal{N})$. Then, a base relation $b \in C(u, v)$, with $u, v \in V(G_i)$, is feasible (resp. unfeasible) iff there exists (resp. there does not exist) a scenario $\mathcal{S}_i = (V_i, C'_i)$ of $\mathcal{N}_i = (V_i, C_i)$ such that $C'_i(u, v) = \{b\}$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow}_{V(G_i)}$, for some $i \in \{1, \ldots, k\}$.*

Regarding the redundancy problem, we can have the following result:

Proposition 8 *Let $\mathcal{N} = (V, C)$ be a satisfiable* QCN *defined on a language that has patchwork for satisfiable atomic* QCN*s, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph $\mathsf{G}(\mathcal{N})$. Then, a relation $C(u, v)$, with $u, v \in V$, is non-redundant in $\mathcal{N}$ iff $(u, v) \in E(G_i)$ and $C(u, v)$ is non-redundant in $\mathcal{N}_i = (V_i, C_i)$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow}_{V(G_i)}$, for some $i \in \{1, \ldots, k\}$.*

Proof Clearly, a relation $C(u, v)$ is redundant in $\mathcal{N}$ if $(v, v') \notin E(G_i)$ for any $i \in \{1, \ldots, k\}$, as it will correspond to the universal relation B, which is by definition redundant. Let us consider a relation $C(u, v)$ where $(u, v) \in E(G_i)$ for some $i \in \{1, \ldots, k\}$. Let $\mathcal{N}' = (V, C')$ be the QCN defined by $C'(u, v) = \mathsf{B} \setminus C(u, v)$, $C'(v, u) = \mathsf{B} \setminus (C(u, v))^{-1}$, and $C'(y, w) = C(y, w)$ $\forall (y, w) \in (V \times V) \setminus \{(u, v), (v, u)\}$. Further, let ${\mathcal{N}_i}' = (V_i, {C_i}')$ be the restriction of $\mathcal{N}'$ to $V(G_i)$. Since $\mathcal{N}$ is satisfiable, $\mathcal{N}'$ is satisfiable, and by Proposition 6 we know that there exists a solution of ${\mathcal{N}_i}'$ and, hence, a scenario of ${\mathcal{N}_i}'$. In addition, as G_i is a biconnected component of $\mathsf{G}(\mathcal{N})$, any scenario $S_i = (V_i, C''_i)$ of ${\mathcal{N}_i}'$ is the restriction of some scenario $S = (V, C'')$ of $\mathcal{N}'$ to V_i, and any scenario $S = (V, C'')$ of $\mathcal{N}'$ is the extension of some scenario $S_i = (V_i, C''_i)$ of ${\mathcal{N}_i}'$ to V. Finally, since for any scenario there exists a solution that satisfies all of its base relations, the redundancy of $C(u, v)$ can be characterized by considering $\mathcal{N}_i$ instead of $\mathcal{N}$. □

Consequently, by Proposition 8 and the fact that IA and RCC-8 have patchwork for satisfiable atomic QCNs of their relations (Lutz and Milicic 2007), we have the following result:

Corollary 7 *Let $\mathcal{N} = (V, C)$ be a satisfiable* QCN *of* IA *or* RCC-*8, and let $\{G_1, \ldots, G_k\}$ be the biconnected components of its constraint graph $\mathsf{G}(\mathcal{N})$. Then, a relation $C(u, v)$, with $u, v \in V$, is non-redundant in $\mathcal{N}$ iff $(u, v) \in E(G_i)$ and $C(u, v)$ is non-redundant in $\mathcal{N}_i = (V_i, C_i)$, where $\mathcal{N}_i$ is $\mathcal{N}{\downarrow}_{V(G_i)}$, for some $i \in \{1, \ldots, k\}$.*

5.1 Dataset

We review the dataset of real RCC-8 network instances that was originally introduced in (Nikolaou and Koubarakis 2014), and which we describe here as follows:

- `nuts`: a nomenclature of territorial units.[10]

[10]Retrieved from: `http://www.linkedopendata.gr/`

- `adm1`: a network that describes the administrative geography of Great Britain (Goodwin, Dolbear, and Hart 2008).
- `gadm1`: a network that describes the German administrative units.[10]
- `gadm2`: a network that describes the world's (global) administrative areas.[11]
- `adm2`: a network that describes the Greek administrative geography.[10]

The aforementioned network instances are tractable and contain at most two base RCC-8 relations per edge. The characteristics of the constraint graphs of these networks are presented in Table 1.

Table 1 Characteristics of real RCC-8 networks

network	# of nodes	# of edges	avg. degree
`nuts`	2 236	3 176	2.84
`adm1`	11 762	44 833	7.62
`gadm1`	42 750	159 600	7.47
`gadm2`	276 728	590 443	4.27
`adm2`	1 733 000	5 236 270	6.04

As it can be seen, the constraint graphs of the networks vary in order, but they are all relatively sparse. This comes as no surprise, as real world graphs often present a scale-free structure (Barabasi and Bonabeau 2003), which results in them being sparse (Del Genio, Gross, and Bassler 2011). Thus, we expect these constraint graphs to be loosely connected and yield a high number of biconnected components. We can view information regarding the biconnected components of the constraint graphs of our networks in Table 2 (where by *max order*, *median order*, and *min order* we refer to the maximum, median, and minimum number of vertices, respectively, met among the biconnected components).

Table 2 Biconnected components of real RCC-8 networks

network	# of components	max order	median order	min order
`nuts`	64	52	8	3
`adm1`	5	11 666	30	3
`gadm1`	166	19 864	6	3
`gadm2`	2 285	2 371	18	3
`adm2`	2 889	22 808	579	4

The findings are quite impressive, in the sense that the maximum order among the biconnected components of a constraint graph is significantly smaller than the order of that graph. For example, the constraint graph of the biggest real RCC-8 network, namely, `adm2`, has an order of value 1 733 000, but the maximum order among its biconnected components is only of value 22 808. Note also that, as the median metric suggests, most of the biconnected components of a graph have an order much closer to the minimum order than the maximum order among the biconnected components of that graph.

Instances for evaluating the satisfiability checking performance of the reasoners for non-tractable QCNs, which are of our interest in this paper, were constructed in (Nikolaou and Koubarakis 2014) with the introduction of $\mathcal{NP}_8$ relations (Renz and Nebel 2001) in the networks' edges. These instances will be denoted by `hard-nuts`, `hard-adm1`, and `hard-gadm1` in the evaluation to follow, and are structurally identical to networks `nuts`, `adm1`, and `gadm1` respectively, i.e., their constraint graphs have the same characteristics as those presented in Tables 1 and 2.

As (Nikolaou and Koubarakis 2014) suggests, some state-of-the-art reasoners, such as GQR (Gantner, Westphal, and Wölfl 2008), use a matrix to represent a QCN $\mathcal{N} = (V, C)$, which has a $O(|V|^2)$ memory requirement. It would be impossible to store a graph of the order of `adm2` in a matrix as we would need $\sim 3TB$ of memory. Even if memory was not the issue, the time complexity alone of a path-consistency algorithm would explode, while the backtracking algorithm that is typically used for tackling non-tractable QCNs and makes use of path-consistency as a forward checking step, would suffer from an increased search space. Heuristics for the backtracking algorithm would also have a hard time distinguishing between biconnected components. Consider

[11]`http://gadm.geovocab.org/`

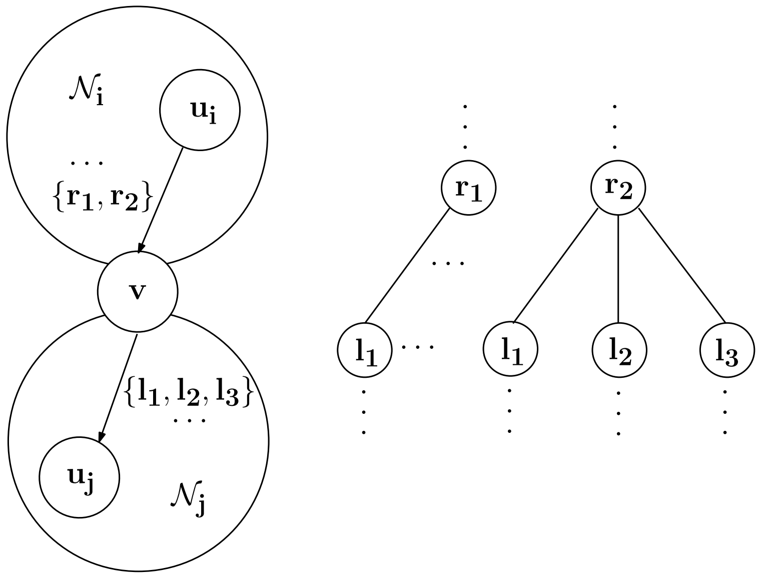


Figure 10 A separable constraint graph with an articulation vertex v

for example a situation where the backtracking algorithm backtracks from an instatiation of a constraint in a biconnected component to an instantiation of a constraint in a different biconnected component. Since the constraints belong to different biconnected components, we have already shown that they are completely unrelated to each other (i.e., satisfying one constraint does not affect the other in any way); nevertheless, they might still define a huge branch in the search-tree that is spawned by the backtracking algorithm. Such a situation is depicted in Figure 10, which presents two QCNs $\mathcal{N}_i = (V_i, C_i)$ and $\mathcal{N}_j = (V_j, C_j)$ such that $V_i \cap V_j = \{v\}$. Let us assume that their constraint graphs are biconnected. Then, the constraint graph of $\mathcal{N}_i \cup \mathcal{N}_j$ has $\mathsf{G}(\mathcal{N}_i)$ and $\mathsf{G}(\mathcal{N}_j)$ as its biconnected components. It is clear that the valuation of constraint $C_i(u_i, v)$ with any of the values r_1 or r_2 does not affect the satisifiability or unsatisfiability of the valuation of constraint $C_j(v, u_j)$ with any of the values l_1, l_2, or l_3, and vice versa. However, if we choose not to treat the biconnected components separately, a huge branch might be defined, as viewed in Figure 10, that could otherwise be entirely avoided. Proposition 6 allows us to treat the QCNs that correspond to biconnected components completely separately, in a parallel or serial fashion, and avoid the aforementioned bothersome issues.

5.2 Evaluation

We consider the *hard* network instances `hard-nuts`, `hard-adm1`, and `hard-gadm1` from (Nikolaou and Koubarakis 2014) that comprise $\mathcal{NP}_8$ relations (Renz and Nebel 2001) to utilize the whole reasoning engine of a reasoner. If Solver is the name of a reasoner, Solver+ denotes the use of Algorithm 4 with that reasoner. The experiments were carried out on a computer with an Intel Core 2 Quad Q9400 processor with a CPU frequency of 2.66 GHz per core, 8 GB RAM, and the Precise Pangolin x86_64 OS. GQR (under version 1500) was compiled with gcc/g++ 4.6.3 and Sarissa, Phalanx, and Phalanx$\bigtriangledown$ (Sioutis and Condotta 2014) (all under version 0.2) were run with PyPy 2.4.0 [12], which fully implements Python 2.7.8. For all reasoners, the best performing heuristics were enabled. (Obviously, we did not consider the implementation of (Nikolaou and Koubarakis 2014) in our evaluation as it is not sound.) We chose to reason in a serial fashion, from smaller to bigger QCN, so as to stress how much more path-consistency and the backtracking algorithm that utilizes it along with the heuristics in each reasoner benefit from reasoning with the smaller biconnected QCNs than reasoning with the initial large and loosely connected QCN, when both approaches are offered the same computational power. Thus, only one CPU core was used in our experiments.

[12]`http://pypy.org`

Table 3 Performance comparison based on elapsed time

solver	GQR	GQR+	Pha.	Pha.+	Sar.	Sar.+	Pha.$\bigtriangledown$	Pha.$\bigtriangledown$+
`hard-nuts`	2.0s	0.1s	4.0s	0.6s	0.8s	0.6s	0.9s	0.6s
`hard-adm1`	4.7E3s	5.2E3s	3.4E3s	3.7E3s	161.5s	137.7s	98.3s	97.4s
`hard-gadm1`	1.4E4s	1.2s	1.0E5s	3.5s	2.0E3s	3.4s	1.1E3s	3.0s

The results are shown in Table 3 and make clear that our simple decomposition scheme aids the performance of each reasoner substantially, with the more apparent case being that of `hard-gadm1`, which is unsatisfiable. Networks `hard-nuts` and `hard-adm1` are satisfiable. In particular, GQR decides `gadm1` in ~ 4 hours, while GQR+ in 1.2 seconds, and similar results are obtained for the other reasoners too. When an inconsistency is detected in a QCN n that corresponds to some biconnected component of the constraint graph of an input QCN $\mathcal{N}$, each reasoner backtracks only within the search space defined by n, and considers a very small search-tree to either verify or dispute that inconsistency with respect to the search-tree that would have been obtained by the input QCN $\mathcal{N}$. Obviously, the time obtained for reasoner Solver+ is the time that it took it to serially reason with every QCN n, until it reached an unsatisfiable QCN (thus, assuring that the input QCN $\mathcal{N}$ is also unsatisfiable by Corollary 5).

It is worth commenting on the performance of the reasoners with respect to network `hard-adm1`. Reasoners Sarissa+ and Phalanx$\bigtriangledown$+ present a performance that is slightly better than that of reasoners Sarissa and Phalanx$\bigtriangledown$ respectively. On the other hand, reasoners GQR+ and Phalanx+ present a performance that is slightly worse than that of reasoners GQR and Phalanx respectively. This is due to the fact that the maximum order among the biconnected components of the constraint graph of `adm1` is very close to the order of the entire graph itself (see Table 2). Thus, in such cases, the use of Algorithm 4 may not lead to drastically improved performance, while sometimes due to the randomness of the heuristics in a reasoner, even slightly worse performance may be observed, as in this particular case.

Finally, we note that the results presented in Table 3 do not take into account the time needed for decomposing the networks with Algorithm 4, but only the time needed for performing satisfiability checks on the networks. However, the time needed for decomposing `hard-nuts`, `hard-adm1`, and `hard-gadm1` was negligible, and does not change the results qualitatively. In particular, a simple Python script that makes use of the networkx[13] library was able to decompose `hard-nuts`, `hard-adm1`, and `hard-gadm1` in 0.2, 1.4, and 7.6 seconds respectively.

6 Related Work

The utility of taking advantage of the structure of qualitative constraint networks has already been addressed in the context of heuristics for the path-consistency algorithms in (Beek and Manchak 1996; Renz 2002b). In particular, the heuristics proposed in (Renz 2002b) target the *denser* parts of the underlying constraint graph of a given qualitative constraint network, i.e., the qualitative relations that consist of few base relations, as an effort to propagate constraints more efficiently and also possibly resolve any local inconsistencies faster. Pruning unfeasible base relations off a qualitative relation that already comprises very few base relations can almost immediately unveil an inconsistency. However, these heuristics always consider a complete underlying constraint graph of a given network. As such, they fail to completely isolate parts of the underlying constraint graph of a given qualitative constraint network that are irrelevant to the process of satisfiability checking; such parts being universal relations that do not belong to the clusters of a tree decomposition corresponding to the constraint graph of the qualitative constraint network at hand. In this paper, we showed that it is possible to omit satisfiability checks across clusters of a tree decomposition corresponding to the constraint graph of a qualitative constraint network.

As noted in our introduction, a qualitative constraint network is most efficiently modeled as an infinite-domain variant of a constraint satisfaction problem through the use of a *relation algebra* (Ladkin and Maddux 1994), which is also the approach we followed in our work.

[13] `https://networkx.github.io/`

However, a qualitative constraint network can also be encoded as a finite constraint satisfaction problem instance (Renz and Nebel 2001; Brand 2004; Condotta et al. 2006). In particular, given a qualitative constraint network (V, C), where $|V| = n$, we can obtain a constraint satisfaction problem instance as follows. Let X denote the set of variables containing a variable x_{ij} for each pair of variables $v_i, v_j \in V$ with $1 \leq i < j \leq n$. Then, our instance has the form $(X, \mathsf{B}, DCon \cup TCon)$, where $DCon$ is the set of domain constraints $\{(x_{ij}, C_{ij}) \mid 1 \leq i < j \leq n\}$ and $TCon$ the set of ternary constraints $\{((x_{ij}, x_{ik}, x_{kj}), R_\diamond) \mid 1 \leq i < j < k \leq n\}$ with $R_\diamond = \{(b, b', b'') \in \mathsf{B}^3 \mid b \in b' \diamond b''\}$. Namely, $DCon$ restricts the values of a variable x_{ij} to the base relations of the corresponding qualitative constraint C_{ij} and $TCon$ encodes all the consistent paths of length 2 in the network. The resulting finite network has $\frac{n(n-1)}{2}$ variables and $\binom{n}{3}$ ternary constraints. A solution of this finite instance corresponds to a path-consistent atomic refinement of a given qualitative constraint network, and vice versa (Condotta et al. 2006). The main disadvantage of this approach is that we are not able to make use of maximal tractable subclasses of relations. This can seriously impact the performance of satisfiability checking for calculi that heavily rely upon those subclasses, such as RCC-8 and IA. However, for large-sized qualitative calculi (viz., comprising hundreds of base relations) for which no tractable subclasses are known, a finite constraint satisfaction problem encoding can provide a considerable performance gain (Westphal and Wölfl 2009).

In light of the strong relation that exists between qualitative and "traditional" constraint programming, it is worth mentioning some works in the latter paradigm that exploit the structure of constraint graphs in a similar manner to what we presented in our paper. The interested reader may review the cited works and obtain a deeper understanding on the analogy that exists between structural characteristics of qualitative constraint networks and finite constraint satisfaction problem instances. What is more important, the cited works may drive future research by enabling the reader to identify theoretical properties in the context of qualitative spatial and temporal reasoning, that can be used to adopt certain techniques for exploiting the structure of constraint graphs that exist in constraint programming. In (Walsh 2001), Walsh measures the impact that the structure of a constraint graph can have on the performance of solving the *graph coloring problem*, which is the problem of coloring the vertices of a graph in such a way that no two adjacent vertices share the same color. In (Baget and Tognetti 2001), Baget et al. propose a backtracking algorithm for solving constraint satisfaction problem instances that exploits the biconnected components of a given constraint graph to reduce search space, permanently removing values and compiling partial solutions during exploitation. In (Dechter and Pearl 1989), Dechter et al. propose a constraint graph restructuring technique, based on tree decompositions, that guarantees that a large variety of queries could be answered swiftly either by sequential backtrack-free procedures, or by distributed constraint propagation methods. Based on the work of (Dechter and Pearl 1989), Jégou et al. in (Jégou and Terrioux 2003) propose a framework for solving constraint satisfaction problem instances that relies both on backtracking techniques and on the notion of tree decomposition of the constraint graphs. Notably, this mixed approach has been implemented and used successfully for practical constraint satisfaction problem solving (Jégou and Terrioux 2003). Jégou et al. in (Jégou, Ndiaye, and Terrioux 2005) study several methods for computing a rough optimal tree decomposition and assess their relevance for solving constraint satisfaction problem instances; the same authors also proposed dynamic heuristics for efficient backtrack search on tree decompositions of constraint graphs in (Jégou, Ndiaye, and Terrioux 2006; Jégou, Ndiaye, and Terrioux 2007). Recently, in (Jégou and Terrioux 2014a; Jégou and Terrioux 2014b) Jégou et al. introduced and exploited a new graph parameter, called *bag-connected tree-width*, which considers tree decompositions for which each cluster induces a connected graph. It is experimentally shown in (Jégou and Terrioux 2014b) that such bag-connected tree decompositions significantly improve the solving of constraint satisfaction problem instances by decomposition methods. Finally, a presentation of the major structural constraint network decomposition methods discussed here is given in (Gottlob, Leone, and Scarcello 2000).

7 Discussion

To conclude, we surveyed the use and effect of decomposition-based techniques in qualitative constraint-based reasoning and showed that the decomposition-based approach presented in (Nikolaou and Koubarakis 2014) for checking the satisfiability of QCNs of RCC-8 lacks soundness, as the notion of a *partitioning graph* defined in that work is not coherent with the use of patchwork upon which it solely relies. Further, we showed how that notion is beyond repair, unless it is reformulated to define a tree decomposition, implicitly or explicitly, and discussed the impact of these observations on the performance of the offered implementation in (Nikolaou and Koubarakis 2014), which was already found to be poor in (Sioutis 2014).

We think that future efforts regarding decomposition-based approaches utilizing parallelism, such as the approach attempted in (Nikolaou and Koubarakis 2014), should rely on chordal graphs (tree decompositions into cliques), which can both be constructed and also yield a natural tree decomposition of their cliques in linear time (Diestel 2012). The cliques can then be collected at no extra cost and parallelism *might* be efficiently utilized. It is an issue that looks promising and calls for further research. Towards that direction, we offered an approach that relies on a particular tree decomposition that is based on the biconnected components of the constraint graph of a given large QCN, and showed that it allows for cost-free utilization of parallelism for a qualitative constraint language that has patchwork for satisfiable atomic QCNs.

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