A Paraconsistency Framework for Inconsistency Handling in Qualitative Spatial and Temporal Reasoning

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Abstract. Inconsistency handling is a fundamental problem in knowledge representation and reasoning. In this paper, we study this problem in the context of qualitative spatio-temporal reasoning, a framework for reasoning about space and time in a symbolic, human-like manner, by following an approach similar to that used for defining paraconsistent logics; paraconsistency allows deriving informative conclusions from inconsistent knowledge bases by mainly avoiding the principle of explosion. Inspired by paraconsistent logics, such as Priest's logic LPm, we introduce the notion of paraconsistent scenario (i.e., a qualitative solution), which can be seen as a scenario that allows a conjunction of base relations between two variables, e.g., x *precedes* $\wedge$ *follows* y. Further, we present several interesting theoretical properties that concern paraconsistent scenarios, including computational complexity results, and describe two distinct approaches for computing paraconsistent scenarios and solving other related problems. Moreover, we provide implementations of our two methods for computing paraconsistent scenarios and experimentally evaluate them using different strategies/metrics. Finally, we show that our paraconsistent scenario notion allows us to adapt to qualitative reasoning one of the well-known inconsistency measures employed in the propositional case, namely, contension measure.

1 Introduction

Inconsistency may arise for many reasons: human error, multi-source information, imprecision and vagueness, noisy data, information evolution over time, etc. This explains the need for inconsistency-tolerant systems to deal with real-world situations. The Knowledge Representation & Reasoning community has extensively studied this topic leading to several inconsistency handling works, e.g., [32, 29, 4, 7, 37, 26, 33]. In this work, we are interested in the use of paraconsistency for inconsistency handling in Qualitative Spatio-Temporal Reasoning (to be introduced in the sequel). A logic is paraconsistent if it does not validate the principle of explosion, which states that any formula can be proven from contradiction. In particular, Priest's minimally inconsistent logic of paradox LPm [27] avoids this principle by allowing variables to be both true and false. This can be seen as a way to allow for the existence of contradictions without collapsing into triviality: consistent and inconsistent elements can coexist in a logical statement without rejecting it as *false*.

In everyday natural language descriptions, one typically uses expressions such as *inside* or *during* to spatially or temporally relate one object with another object or oneself, without providing the exact metric information about these entities. An AI framework that aims to capture this type of human-like representation and reasoning pertaining to space and time is known as Qualitative Spatio-Temporal Reasoning (QSTR) [15, 25]. Specifically, QSTR is a major field of study in Knowledge Representation and Reasoning that deals with the concepts of space and time in an abstract, natural manner, with applications in many areas such as visual sensemaking [36] and qualitative case-based reasoning and learning [19], to name some recent ones. More formally, QSTR restricts the vocabulary of rich mathematical theories that deal with spatial and temporal entities to simple qualitative constraint languages, which can be used to form interpretable spatio-temporal constraint networks of disjunctions of base relations, such as the one shown in Figure 1. However, as with any other typical logic- or constraint-based framework, QSTR is not immune to contradictions that may be present in information.

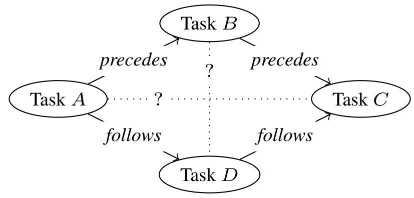

Figure 1. A simplified temporal constraint network, where ? denotes the *disjunction* of all base relations; the constraint between Tasks B and D (and, equivalently, A and C) is not repairable, but we can replace ? with the *conjunction* '*precedes* $\wedge$ *follows*' and achieve paraconsistency.

Motivation

Naturally, the motivation behind studying paraconsistency in the context of QSTR draws from the rich literature in paraconsistency itself. However, we present an example here to help the reader understand one of many cases of how this notion can apply to QSTR. Consider Figure 1, and let us ground it in a realistic scenario of task scheduling in a factory. We can view Task D as an *inspection* task of a product in the production pipeline, and the other tasks as necessary components in the manufacturing process of that product. Clearly, a mistake occured in the design of the pipeline, as the schedule is unfeasible. The constraint between Tasks B and D is not repairable, so, to restore consistency, we would have to repair some other constraints. However, this may be impossible too, due to hard dependencies in

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the pipeline, e.g., product preparation, say Task B, *precedes* product packaging, say Task C. Instead of rejecting the entire schedule, we opt to acknowledge the contradiction and reason with it. Here, we can retrieve a paraconsistent configuration where Task B both *precedes and follows* Task D. This not only helps us to understand the contradiction, but, in this example, to also observe that the *inspection* task was probably meant to occur both at an earlier and at a later stage of the production pipeline and should thus be replicated.

Contributions

Our *first* contribution is the introduction of the notion of paraconsistent scenario (para-scenario for short). To some extent, it can be seen as an adaptation of the approach used for defining LPm to QSTR. Indeed, similarly to LPm, where an interpretation can assign a propositional variable more than one truth value, our base idea consists in allowing constraints to involve a conjunction of more than one base relation, as a means to achieve compatibility with other constraints (see Figure 1); then, we focus on the para-scenarios that are as consistent as possible, which are obtained by avoiding as much as possible the use of such conjunctions.

Our *second* contribution is the theoretical study of several interesting properties of para-scenarios. In particular, we show that the problem of determining whether an interpretation is a para-scenario is coNP-complete in the case of several well-known QSTR formalisms.

Our *third* contribution involves providing and evaluating two open-source approaches for solving the problem of para-scenario computation and other related problems. The first approach is based on a notion of constraint freezing (cf. [10]) within calls to a native qualitative reasoner: when a constraint is frozen, it cannot lose any base relation during solving, but, in contrast to [10], it can participate in compositions with other constraints. The second approach consists of using SAT-based encodings, where we involve the problems of X-minimal model computation and Partial MaxSAT in particular.

Finally, our *fourth* contribution is showing how the notion of para-scenario can be used for inconsistency measurement. Indeed, we propose inconsistency measures that can be seen as the first adaptation of the well-known contension measure to QSTR [16]. This contribution is provided just as a concrete example of how our framework can be exploited to analyze/measure inconsistency. The definition of such measures in the literature is commonly guided by rationality postulates. In our study, we show that our measures fulfill postulates that enjoy a broad consensus.

Organization The rest of the paper is organized as follows. In Section 2 we recall some definitions, and introduce certain necessary notions and notations that are used throughout the paper. Then, in Section 3 we introduce and theoretically establish our paraconsistency framework for QSTR, and prove certain fundamental properties. Next, in Section 4 we provide greedy constraint-based and optimal SAT-based methods for solving various paraconsistency-related problems, which we proceed to evaluate. Moreover, in Section 5 we propose some inconsistency measures pertaining to paraconsistency, and consequently show how our ideas can be used to quantify inconsistency in QSTR. Finally, in Section 6 we conclude our work and give some directions for future research.

2 Preliminaries and Notations

A binary qualitative spatial or temporal constraint language is based on a finite set B of *jointly exhaustive and pairwise disjoint* relations,

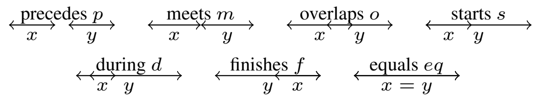

Figure 2. A representation of the 13 base relations b of IA, each one relating two potential intervals x and y as in $x\ b\ y$; the converse of b, i.e., b^{-1}, can be denoted by bi and is omitted in the figure.

called *base relations* [25] and defined over an infinite domain D (e.g., $\mathbb{R}$). Specifically, $\bigcup\{b \in \mathsf{B}\} = \mathsf{D} \times \mathsf{D}$, and, $\forall b, b' \in \mathsf{B}$ s.t. $b \neq b'$, $b \cap b' = \emptyset$. The base relations of a particular qualitative constraint language can be used to represent the definite knowledge between any two of its entities with respect to the level of granularity provided by the domain D. The set B contains the identity relation Id, and is closed under the *converse* operation ($^{-1}$). Indefinite knowledge can be specified by a union of possible base relations, and it is represented by the set containing them. Hence, 2^{B} represents the total set of relations. The set 2^{B} is equipped with the usual set-theoretic operations of union and intersection, the converse operation, and the *weak composition* operation denoted by the symbol $\diamond$ [25]. For all $r \in 2^{\mathsf{B}}$, we have that $r^{-1} = \bigcup\{b^{-1} : b \in r\}$. The weak composition ($\diamond$) of two base relations $b, b' \in \mathsf{B}$ is defined as the smallest (i.e., most restrictive) relation $r \in 2^{\mathsf{B}}$ that includes $b \circ b'$, or, formally, $b \diamond b' = \{b'' \in \mathsf{B} : b'' \cap (b \circ b') \neq \emptyset\}$, where $b \circ b' = \{(x, y) \in \mathsf{D} \times \mathsf{D} : \exists z \in \mathsf{D}$ such that $(x, z) \in b \wedge (z, y) \in b'\}$ is the (true) composition of b and b'. For all $r, r' \in 2^{\mathsf{B}}$, we have that $r \diamond r' = \bigcup\{b \diamond b' : b \in r, b' \in r'\}$.

As an illustration, consider the well-known qualitative temporal constraint language of Interval Algebra (IA), introduced by Allen [1]. IA considers time intervals (as temporal entities) and the set of base relations $\mathsf{B} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$ to encode knowledge about the temporal relations between intervals on the timeline, as described in Figure 2. As another example, the Region Connection Calculus (RCC8) [28] considers spatial regions and the set of base relations $\mathsf{B} = \{DC, EC, EQ, PO, TPP, TPPi, NTPP, NTPPi\}$ to reason about topological relations between regions. Regarding RCC5, it is a fragment of RCC8 where boundaries of regions have no significance [5].

Finally, the challenge of representing and reasoning about qualitative spatio-temporal information can be facilitated by a *qualitative spatio-temporal network (QCN)*, defined as follows:

Definition 1 *A* qualitative spatio-temporal network (QCN) *is a tuple* (V, C) *where:*

- V *is a finite set of variables over some infinite domain* D *(e.g., time points or 2D regions);*
- *and* C *is a mapping* $C : V \times V \to 2^{\mathsf{B}}$ *associating a relation (set of base relations) with each pair of variables s.t.* $C(v, v) = \{Id\}$ *for all* $v \in V$ *and* $C(v, v') = (C(v', v))^{-1}$ *for all* $v, v' \in V$.

A simplified QCN is shown in Figure 1. For convenience, we often consider that the set of variables of a QCN consists of integers.

A QCN $\mathcal{N} = (V, C)$ is said to be *trivially inconsistent* iff $\exists v, v' \in V$ such that $C(v, v') = \emptyset$.

A *solution* of a QCN $\mathcal{N} = (V, C)$ is a mapping $\sigma : V \to \mathsf{D}$ such that $\forall v, v' \in V$, $\exists b \in C(v, v')$ such that $(\sigma(v), \sigma(v')) \in b$; $\mathcal{N}$ is said to be *consistent* if and only if it admits a solution.

A *sub-QCN* $\mathcal{N}'$ of $\mathcal{N}$, denoted by $\mathcal{N}' \subseteq \mathcal{N}$, is a QCN (V, C') such that, $\forall u, v \in V$, $C'(u, v) \subseteq C(u, v)$.

A *scenario* of $\mathcal{N}$ is a consistent atomic sub-QCN $\mathcal{S}$ of $\mathcal{N}$, where a QCN $\mathcal{S} = (V, C')$ is *atomic* iff, $\forall v, v' \in V$, $|C'(v, v')| = 1$.

A *max-scenario* of $\mathcal{N}$ is a consistent atomic QCN $\mathcal{S} = (V, C')$ s.t. there is no other consistent atomic QCN $\mathcal{S}' = (V, C'')$ s.t. $\{(v, v') \in V \times V : C'(v, v') \subseteq C(v, v')\} \subsetneq \{(v, v') \in V \times V : C''(v, v') \subseteq C(v, v')\}$. Roughly speaking, a max-scenario of $\mathcal{N}$ maximizes the number of satisfied constraints in $\mathcal{N}$. Here, given a max-scenario $\mathcal{S} = (V, C')$ of $\mathcal{N}$, we use $size(\mathcal{S})$ to denote the size of the set $\{(v, v') \in V \times V : v < v', C'(v, v') \subseteq C(v, v')\}$. Furthermore, we use $MCS(\mathcal{N})$ to denote the set of all max-scenarios of $\mathcal{N}$.

The following notational conventions are used throughout the paper for every QCN $\mathcal{N} = (V, C)$:

- For two variables $v, v' \in V$, we use $\mathcal{N}[v, v']$ to denote the relation $C(v, v')$.
- For two variables $v, v' \in V$ and a relation $r \in 2^{\mathsf{B}}$, we use $v\ r\ v'$ to denote that $C(v, v') = r$ when there is no ambiguity about the considered QCN.
- For two variables $v, v' \in V$ and a relation $r \in 2^{\mathsf{B}}$, we use $\mathcal{N}_{[v,v']/r}$ to denote the result of substituting $C(v, v')$ with r in $\mathcal{N}$. Formally, $\mathcal{N}_{[v,v']/r}$ is the QCN (V, C') defined by $C'(v, v') = r$, $C'(v', v) = r^{-1}$ and, $\forall (u, u') \in (V \times V) \setminus \{(v, v'), (v', v)\}$, $C'(u, u') = C(u, u')$.
- For $V' \subseteq V$, we use $\mathcal{N}{\downarrow}_{V'}$ to denote the QCN $\mathcal{N}$ restricted to V'.

Finally, for the sake of simplicity, it is worth noting that our focus in this work is exclusively on calculi in which every atomic QCN that is closed under weak composition is consistent, which covers the majority of well-known calculi [15].

3 Para-scenarios and Paraconsistency

We introduce a new notion called paraconsistent scenario (or para-scenario for short), which extends the idea of a scenario to include multiple base relations (i.e., a conjunction of them) between two variables. This notion is analogous to that of minimally inconsistent model in the paraconsistent logic LPm [27], where a variable can take multiple truth values by being both true and false. Intuitively, the notion of para-scenario allows us to take into account the possible conflicts without considering that the whole QCN is not informative.

We define a *para-relation* as an expression of the form $b_1 \wedge \cdots \wedge b_k$ where $k \geq 1$ and $b_1, \ldots, b_k$ are distinct base relations. We use $\mathcal{C}onj(\mathsf{B})$ to denote the set of para-relations built over the set of base relations B. For the sake of simplicity, we often refer to a para-relation as a non-empty set of base relations. Given two para-relations c and c', we use $c \diamond c'$ to denote the para-relation $\bigwedge \bigcup_{b \in c, b' \in c'} b \diamond b'$. Furthermore, we write $(b_1 \wedge \cdots \wedge b_k)^{-1}$ for $(b_1)^{-1} \wedge \cdots \wedge (b_k)^{-1}$.

Definition 2 (Paraconsistent Scenario) *A* paraconsistent scenario *(or para-scenario for short) of a QCN $\mathcal{N} = (V, C)$ is an ordered pair $\mathcal{P} = (V, R)$ where R is a mapping $V \times V \to \mathcal{C}onj(\mathsf{B})$ s.t.:*

1. *(Identity) for all $v \in V$, $R(v, v) = \mathsf{id}$;*
2. *(Conversion) for $v, v' \in V$, $R(v, v') = (R(v', v))^{-1}$;*
3. *(Paraconsistency) for all $v, v' \in V$, $R(v, v') \cap C(v, v') \neq \emptyset$;*
4. *(Closedness) for all $v, v', v'' \in V$, there exist $b \in R(v, v'')$, $b' \in R(v, v')$ and $b'' \in R(v', v'')$ s.t. $b \in b' \diamond b''$; and*
5. *(Minimality) there is no $\mathcal{P}' = (V, R')$ that satisfies Properties 1-4, and $\{(v, b, v') \in V \times \mathsf{B} \times V : |R'(v, v')| > 1$ and $b \in R'(v, v')\} \subsetneq \{(v, b, v') \in V \times \mathsf{B} \times V : |R(v, v')| > 1$ and $b \in R(v, v')\}$.*

Given a para-scenario $\mathcal{P} = (V, R)$, the notation $\mathcal{P}!$ represents the set $\{(i, b, j) \in V \times \mathsf{B} \times V : i < j, |R(i, j)| > 1$ and $b \in R(i, j)\}$.

The *paraconsistency degree* of para-scenario $\mathcal{P} = (V, R)$, denoted $\Delta(\mathcal{P})$, is the number of pairs of variables where the para-relation contains more than one base relation: $\Delta(\mathcal{P}) = \{\{i, j\} : i, j \in V, |R(i, j)| > 1\}$. The *paraconsistency width* of $\mathcal{P} = (V, R)$, denoted $W(\mathcal{P})$, is the maximum number of base relations between two variables occurring in $\mathcal{P}$: $W(\mathcal{P}) = max\{|R(i, j)| : i, j \in V\}$.

Proposition 1 *Every not trivially inconsistent QCN $\mathcal{N}$ admits a para-scenario $\mathcal{P}$ where $W(\mathcal{P}) \leq 2$ and $\Delta(\mathcal{P}) \leq k$ where $k = min\{m - size(\mathcal{S}) : \mathcal{S} \in MCS(\mathcal{N})\}$ and $m = |\{(v, v') \in V \times V : v < v'\}|$.*

Proof. Let $\mathcal{N} = (V, C)$ be a not trivially inconsistent QCN and $\mathcal{S} = (V, C')$ a max-scenario of $\mathcal{N}$ s.t. $size(\mathcal{S}) = m - k$. Then, we define $\mathcal{P} = (V, R)$ where R is a mapping $V \times V \to \mathcal{C}onj(\mathsf{B})$ as follows: $\forall i, j \in V$ with $\mathcal{S}[i, j] \subseteq C(i, j)$, $R(i, j) = \mathcal{S}[i, j]$; $\forall i, j \in V$ with $i < j$ and $\mathcal{S}[i, j] \not\subseteq C(i, j)$, $R(i, j) = \bigwedge \mathcal{S}[i, j] \cup \{b\}$ and $R(j, i) = (R(i, j))^{-1}$, where b is an arbitrary base relation occurring in $C(i, j)$. Clearly, $\mathcal{P}$ satisfies Properties 1-4 in Definition 2. Further, $max\{|R(i, j)| : i, j \in V\} \leq 2$ and $\Delta(\mathcal{P}) = k$. Thus, due to the definition of Minimality, we deduce that there exists a para-consistent scenario of $\mathcal{N}$ where $W(\mathcal{P}) \leq 2$ and $\Delta(\mathcal{P}) \leq k$. ⊣

The following proposition shows that the notion of para-scenario allows recovering classical reasoning in the case of consistent QCNs.

Proposition 2 *If $\mathcal{N}$ is a consistent QCN, then $\mathcal{P}$ is a para-scenario of $\mathcal{N}$ iff $\mathcal{P}$ is a scenario of $\mathcal{N}$.*

Proof. This property is mainly due to the fact that $\mathcal{P}! = \emptyset$ iff $\mathcal{P}$ is a consistent atomic QCN. ⊣

Given a QSTR formalism $\mathcal{F}$, we use $\mathsf{IsPS}(\mathcal{F})$ to refer to the following problem:

- **Input:** A QCN $\mathcal{N}$ in $\mathcal{F}$ and an ordered pair $\mathcal{P} = (V, R)$ where R is a mapping $V \times V \to \mathcal{C}onj(\mathsf{B})$ that satisfies Properties 1-4 in Definition 2.
- **Output:** Determine whether $\mathcal{P}$ is a para-scenario of $\mathcal{N}$.

Theorem 1 *The problems $\mathsf{IsPS}(\mathsf{IA})$, $\mathsf{IsPS}(\mathsf{RCC5})$ and $\mathsf{IsPS}(\mathsf{RCC8})$ are coNP-complete.*

Proof. Let us first show that $\mathsf{IsPS}(\mathcal{F})$ are in coNP for $\mathcal{F} \in \{\mathsf{IA}, \mathsf{RCC5}, \mathsf{RCC8}\}$. Let $\mathcal{N}$ be a QCN and $\mathcal{P} = (V, R)$ an ordered pair s.t. R is a mapping $V \times V \to \mathcal{C}onj(\mathsf{B})$ that satisfies Properties 1-4 of the definition of para-scenario. To show that $\mathcal{P}$ is not a para-scenario of $\mathcal{N}$, we only need to show that there exists $\mathcal{P}' = (V, R')$ s.t. R' satisfies Properties 1-4 and $\mathcal{P}'! \subsetneq \mathcal{P}!$. Thus the proof that $\mathcal{P}$ is not a para-scenario is verifiable in polynomial time, which yields that the complement problem of $\mathsf{IsPS}(\mathcal{F})$ is in NP. Consequently, $\mathsf{IsPS}(\mathcal{F})$ is in coNP.

We now show that $\mathsf{IsPS}(\mathsf{IA})$, $\mathsf{IsPS}(\mathsf{RCC5})$ and $\mathsf{IsPS}(\mathsf{RCC8})$ are coNP-hard.

Case of IA. To prove that $\mathsf{IsPS}(\mathsf{IA})$ is coNP-hard, we provide an encoding of the 3-coloring problem into the complement of $\mathsf{IsPS}(\mathsf{IA})$. Let $G = (V, E)$ be an undirected graph with $V = 1, \ldots, n$. To define our encoding we consider that each element of V is an interval variable. Furthermore, we associate an additional interval variable x_i with each $i \in V$; the set of these variables is denoted V'. Then our encoding $\mathcal{N}_G$ corresponds to the following constraints:

$$i\ \{m, eq, mi\}\ x_i \text{ for every } i \in V \tag{1}$$

$$i \, \mathsf{B} \, x_j \text{ for every } i, j \in V \text{ with } i \neq j \quad (2)$$

$$i \, \{m, mi, p, pi\} \, j \text{ for every } \{i, j\} \in E \quad (3)$$

$$x_i \, \{p, eq\} \, x_j \text{ for every } i, j \in V \text{ with } i < j \text{ and } i \neq 1 \text{ or } j \neq n \quad (4)$$

$$x_1 \, \{eq\} \, x_n \quad (5)$$

The QCN $\mathcal{N}_G$ corresponds to an encoding of the 3-coloring problem for the instance G; a solution is obtained by associating a distinct color with each vertex i depending on which of the following constraints is satisfied: $i\{mi\}x_i$, $i\{eq\}x_i$, and $i\{m\}x_i$. Indeed, for every $\{i, j\} \in E$, if $i\{b\}x_i$ with $b \in \{m, eq, mi\}$, then, using Constraint (3), we have $j\{b'\}x_i$ with $b' \in \{m, eq, mi\} \setminus \{b\}$.

Let $\mathcal{P} = (V \cup V', R)$ where R is a mapping $V \times V \to \mathcal{C}onj(\mathsf{B})$ that satisfies Identity and Conversion, and is defined as follows: $R(i, x_i) = \{eq\}$ for every $i \in V$; $R(i, j) = R(i, x_j) = \{p\}$ for all $i, j \in V$ with $i < j$; $R(x_i, x_j) = \{p\}$ for all $i, j \in V$ with $i < j$, and $i \neq 1$ or $j \neq n$; $R(x_1, x_n) = \{eq, p\}$. Clearly $\mathcal{P}$ satisfies Paraconsistency and Closedness. Moreover, $\mathcal{P}$ does not satisfy Minimality if and only if $\mathcal{N}_G$ is satisfied (see Proposition 2). Therefore, $\mathcal{P}$ satisfies Minimality if and only if G does not admit a 3-coloring.

Case of RCC5. Sketch: To show coNP-hardness in this case, we also use the complement problem of 3-coloring. Additionally, we use Renz and Nebel's encoding of the NP-hard problem Not-All-Equal-3SAT (NAE-3SAT) into RCC5 [30]. First, we encode the 3-coloring problem into NAE-3SAT. Then, the obtained formula is encoded into RCC5 using Renz and Nebel's encoding, and we exhibit a para-scenario $\mathcal{P}$ candidate of this encoding such that $|\mathcal{P}!| = 1$. Thus, using again Proposition 2, $\mathcal{P}$ satisfies Minimality if and only if the RCC5 encoding is inconsistent.

Case of RCC8. The coNP-hardness of IsPS(RCC5) is a direct consequence of the coNP-hardness of IsPS(RCC5). $\dashv$

Note that a para-scenario may involve base relations between two variables that do not occur in the corresponding constraint. This can be problematic, especially when any such base relation that does not appear between two variables is considered as impossible and must be avoided in any information retrieval process. To address this problem, we introduce the notion of strong para-scenario.

Definition 3 (Strong Paraconsistent Scenario) *A para-scenario $\mathcal{P} = (V, R)$ of a QCN $\mathcal{N}$ is said to be* strong *if for all variables $i, j \in V$, $R(i, j) \subseteq C(i, j)$.*

Clearly, there are not trivially inconsistent QCNs that do not admit strong para-scenarios. This is particularly true for all inconsistent atomic QCNs.

Proposition 3 *The problem of determining whether a QCN admits a strong para-scenario is tractable.*

Proof. Given a QCN $\mathcal{N} = (V, C)$, we only need to check whether $\mathcal{P} = (V, R)$ satisfies Properties 1-4 in Definition 2, where for all $i, j \in V$, $R(i, j) = \bigwedge C(i, j)$. This comes form the fact that $\mathcal{P}$ satisfies these properties iff there exists a para-scenario $\mathcal{P}'$ of $\mathcal{N}$ s.t. $\mathcal{P}'! \subseteq \mathcal{P}!$. $\dashv$

4 On Computing Para-scenarios

There are several interesting problems to study related to the notion of para-scenario. In particular, the search for one or all para-scenarios can be involved in the definition of paraconsistent consequence relations, which can be employed for extracting knowledge from inconsistent QCNs. For instance, a paraconsistent consequence relation $\vdash_p$ can be defined as follows: $\mathcal{N} \vdash_p i \; r \; j$ holds iff for every (strong) para-scenario $\mathcal{P} = (V, R)$ of $\mathcal{N}$, $r \cap R(i, j) \neq \emptyset$; in other words, a relation between two variables is accepted if it shares at least one base relation with the para-relation between these variables in each para-scenario. Other problems may involve the search for para-scenarios that approximate classical reasoning as closely as possible, which can be assessed using measures such as the paraconsistency degree, the paraconsistency width and the size of the para-relations that contain more than one base relation. We here consider the five following problems, the last three of which are optimization problems:

- FindOnePS

  Input: A QCN $\mathcal{N}$ and 2 positive integers α and β.

  Output: A para-scenario $\mathcal{P}$ of $\mathcal{N}$ that satisfies $\Delta(\mathcal{P}) \leq \alpha$ and $W(\mathcal{P}) \leq \beta$ if there exists at least one.

- FindAllPS

  Input: A QCN $\mathcal{N}$ and 2 positive integers α and β.

  Output: All para-scenarios $\mathcal{P}$ of $\mathcal{N}$ such that $\Delta(\mathcal{P}) \leq \alpha$ and $W(\mathcal{P}) \leq \beta$.

- MinimumDegreePS

  Input: A QCN $\mathcal{N}$ and a positive integer β.

  Output: A para-scenario $\mathcal{P}$ of $\mathcal{N}$ where $W(\mathcal{P}) \leq \beta$, and for every para-scenario $\mathcal{P}'$ of $\mathcal{N}$, $\Delta(\mathcal{P}) \leq \Delta(\mathcal{P}')$.

- MinimumWidthPS

  Input: A QCN $\mathcal{N}$ and a positive integer α.

  Output: A para-scenario $\mathcal{P}$ of $\mathcal{N}$ where $\Delta(\mathcal{P}) \leq \alpha$, and for every para-scenario $\mathcal{P}'$ of $\mathcal{N}$, $W(\mathcal{P}) \leq W(\mathcal{P}')$.

- MinimumBaseRelPS

  Input: A QCN $\mathcal{N}$.

  Output: A para-scenario $\mathcal{P}$ of $\mathcal{N}$ where for every para-scenario $\mathcal{P}'$ of $\mathcal{N}$, $|\mathcal{P}!| \leq |\mathcal{P}'!|$.

4.1 A Greedy Constraint Freezing-based Approach

We first provide a procedure, described in Algorithm 1, that solves the problem FindOnePS and greedily aims to come close to solutions for the optimizations problems MinimumDegreePS, MinimumWidthPS, and MinimumBaseRelPS.

Definition 4 (F-Paraconsistent Scenario) *Let $\mathcal{N} = (V, C)$ be a QCN and F a subset of $\{(i, j) \in V \times V : i < j\}$. A F-*paraconsistent scenario *of $\mathcal{N}$ is an ordered pair $\mathcal{P} = (V, R)$ that satisfies (i) Properties 1-4 in Definition 2; (ii) for every $(i, j) \in F$, $R(i, j) = \bigwedge_{b \in C(i,j)} b$; and (iii) for every $(i, j) \in (V \times V) \setminus F$ with $i < j$, $|R(i, j)| = 1$.*

We say that the constraints that concern the pairs in F are *frozen*.

Proposition 4 *Let $\mathcal{N} = (V, C)$ be a QCN. If $\mathcal{P} = (V, R)$ is an F-paraconsistent scenario of a QCN $\mathcal{N}'$ s.t.*

1. *$\forall i, j \in V$ with $i < j$, if $(i, j) \in F$ then $\mathcal{N}'[i, j] = \mathsf{B}$;*
2. *$\forall i, j \in V$ with $i < j$, if $(i, j) \notin F$ then $\mathcal{N}'[i, j] = \mathcal{N}[i, j]$;*

Algorithm 1: FINDPARASCENARIO

Data: A QCN $\mathcal{N} = (V, C)$
Result: A para-scenario of $\mathcal{N}$
Let $\mathcal{S}_0$ be an arbitrary consistent atomic QCN
$\mathcal{P} \leftarrow \mathcal{N}$;
for $i, j \in V$ *with* $i < j$ **do**
  if $\mathcal{S}_0[i, j] \not\subseteq C(i, j)$ **then**
    $\mathcal{P}[i, j] \leftarrow \mathsf{B}$;
    $frozenCons[i, j] \leftarrow true$;
  else
    $frozenCons[i, j] \leftarrow false$;
for $i, j \in V$ *with* $i < j$ *and* $frozenCons[i, j] = true$ **do**
  if $SAT(\mathcal{P}_{[i,j]/\mathcal{N}[i,j]}, frozenCons_{[i,j]/false})$ **then**
    $frozenCons[i, j] \leftarrow false$;
    $\mathcal{P}[i, j] \leftarrow \mathcal{N}[i, j]$;
for $i, j \in V$ *with* $i < j$ *and* $frozenCons[i, j] = true$ **do**
  $M \leftarrow \{(i, b, j) \ : \ b \in \mathsf{B}\}$;
  while $|\mathcal{P}[i, j]| > 2$ *and* $\exists (i, b, j) \in M$ *s.t.* $C(i, j) \cap (\mathcal{P}[i, j] \setminus \{b\}) \neq \emptyset$ **do**
    $r \leftarrow \mathcal{P}[i, j] \setminus \{b\}$;
    if $SAT(\mathcal{P}_{[i,j]/r}, frozenCons)$ **then**
      $\mathcal{P}[i, j] \leftarrow r$;
    $M \leftarrow M \setminus \{(i, b, j)\}$;
return The last found F-paraconsistent scenario

3. *$\forall F' \subsetneq F$, there is no F'-paraconsistent scenario of a QCN $\mathcal{N}''$ that satisfies Properties 1 and 2 w.r.t. F';*

then there exits a para-scenario $\mathcal{P}' = (V, R')$ of $\mathcal{N}$ s.t. $\{(i, j) \in V \times V : i < j, |R'(i, j)| > 1\} = F$.

Proof. Using the fact that $\mathcal{P}$ satisfies Properties 1-4 in Definition 2, we know that there exits a para-scenario $\mathcal{P}' = (V, R')$ of $\mathcal{N}$ s.t. $F' = \{(i, j) \in V \times V : i < j, |R'(i, j)| > 1\} \subseteq F$. Let us first define the QCN $\mathcal{N}'' = (V, C'')$ as follows: for all $i, j \in V$ with $i < j$, if $(i, j) \in F$ then $C''(i, j) = \mathsf{B}$; otherwise, $C''(i, j) = C(i, j)$. Then, we define $\mathcal{P}'' = (V, R'')$ as follows: for all $i, j \in V$ with $i < j$, if $(i, j) \in F'$ then $R''(i, j) = \mathsf{B}$; otherwise, $R''(i, j) = R'(i, j)$. Clearly, $\mathcal{P}''$ is an F'-paraconsistent scenario of $\mathcal{N}''$. Consequently, due to Property 3, if we assume $F' \subsetneq F$, we get a contradiction. $\dashv$

In the first for-loop of our procedure FINDPARASCENARIO, we compute a starting F-paraconsistent scenario where every frozen constraint contains all base relations. Then, in the second for-loop, we unfreeze the constraints that are unnecessarily frozen: this allows us to minimize the set of frozen constraints w.r.t. set inclusion. The last for-loop is used to reduce the number of base relations to satisfy Minimality. For convenience, $\mathcal{P}[i, j] \leftarrow r$ also assumes $\mathcal{P}[j, i] \leftarrow (r)^{-1}$, and $\forall i \in V$, $\mathcal{P}[i, i] \leftarrow \mathsf{Id}$. The SAT calls are *almost* standard with regard to any native qualitative reasoner (e.g., [39]), the only difference being that the solving process is modified to protect frozen constraints from getting refined. Specifically, our intervention is very similar to the one described in [10, Section 5], but, contrary to that work, (i) we allow frozen constraints to be selected and participate in constraint propagation, and (ii) we perform a closedness check in each SAT call for triangles of frozen non-universal constraints (should they exist); this is because in our case the set of frozen constraints is not necessarily consistent as in [10].

The soundness of our algorithm can be seen from mainly Proposition 4. Indeed, the first two for-loops in FINDPARASCENARIO compute an F-paraconsistent scenario of a QCN that fulfills the three specified properties. Thus, we know that there exists a para-scenario $\mathcal{P} = (V, R)$ of $\mathcal{N}$ s.t. $\{(i, j) \in V \times V : i < j, |R'(i, j)| > 1\} = F$. In the last for-loop, by taking each frozen constraint one by one, we remove the unnecessary base relations while satisfying the unfrozen constraints. This clearly leads to a para-scenario.

4.2 An Optimal SAT-based Approach

We first solve the problem FindAllPS (which, of course, subsumes FindOnePS) using the notion of X-minimal model.

Definition 5 (X-minimal Model) *Let ϕ be a propositional formula and X a subset of propositional variables. An X-minimal model of ϕ is a model ω of ϕ s.t. there is no model ω' of ϕ s.t. $\{p : \omega'(p) = 1\} \subsetneq \{p : \omega(p) = 1\}$.*

Specifically, we define an encoding for generating the para-scenarios of a given QCN $\mathcal{N} = (V, C)$ based on the notion of X-minimal model. To this end, we associate two propositional variables p^b_{ij} and q^b_{ij} with each base relation $b \in \mathsf{B}$ and each pair of variables $i, j \in V$ s.t. $i < j$. The variables of the form p^b_{ij} are used to represent the para-scenarios, whereas those of the form q^b_{ij} are used to satisfy Minimality. We also associate with each pair of variables i and j with $i < j$ a propositional variable r_{ij}, which is used to know whether or not there is more than one base relation between i and j. Additionally, we use Comp to refer to the set $\{(b, b', b'') \in \mathsf{B} \times \mathsf{B} \times \mathsf{B} : b \in b' \diamond b''\}$.

Our first formula guarantees the satisfaction of Paraconsistency:

$$\bigwedge_{i,j \in V, i<j} \Big(\bigvee_{b \in C(i,j)} p^b_{ij} \Big) \tag{6}$$

The second formula is used to satisfy Closedness:

$$\bigwedge_{i,j,k \in V, i<j<k} \ \bigvee_{(b,b',b'') \in \mathsf{Comp}} (p^b_{ik} \wedge p^{b'}_{ij} \wedge p^{b''}_{jk}) \tag{7}$$

The following formula allows us to identify the pairs of variables that involve more than one base relation; if r_{ij} is false, it indicates that the corresponding relation is not paraconsistent and consists of only one base relation:

$$\bigwedge_{i,j \in V, i<j} \ \bigwedge_{b \in C(i,j), b' \in \mathsf{B}, b \neq b'} (p^b_{ij} \wedge p^{b'}_{ij} \rightarrow r_{ij}) \tag{8}$$

Through the notion of X-minimal model, the formula presented below ensures Minimality by minimizing the variables assigned 1 among the variables of the form q^b_{ij}:

$$\bigwedge_{i,j \in V, i<j} \ \bigwedge_{b \in \mathsf{B}} (p^b_{ij} \wedge r_{ij} \rightarrow q^b_{ij}) \tag{9}$$

The following two formulas allows us to take into account the bounds on the paraconsistency degree and the paraconsistency width of the found para-scenarios, respectively:

$$\sum_{i,j \in V, i<j} r_{ij} \leq \alpha \tag{10}$$

$$\sum_{b \in \mathsf{B}} p^b_{ij} \leq \beta \text{ for every } i, j \in V, i < j \tag{11}$$

Let us note that the subformulas of the form $\sum_{i=1}^{n} x_i \leq m$ correspond to the well-known *cardinality constraints*. Various polynomial encodings of these constraints into propositional formulas have been proposed in the literature (e.g., see [34]).

We use $\mathsf{Enc}_1(\mathcal{N}, \alpha, \beta)$ to denote the formula that corresponds to the conjunction of Formulas (6)–(11).

Algorithm 2: GREEDYREPAIR

Data: A QCN $\mathcal{N} = (V, C)$
Result: A consistent QCN $\mathcal{N}'$ with $\mathcal{N}' \supseteq \mathcal{N}$

$\mathcal{N}' \leftarrow \mathcal{N}$;
$(i, j) \leftarrow SAT(\mathcal{N}')$;   `//` (i,j) `with` $i < j$ `for a violated constraint` $\mathcal{N}'[i,j]$`, if it exists`
if $\nexists(i, j)$ **then return** $\mathcal{N}'$;
$repairs \leftarrow \{(i, j)\}$;
while $true$ **do**
  Let $(i, j) \in repairs$;
  $\mathcal{N}'[i, j] \leftarrow \mathsf{B}$;
  $(i', j') \leftarrow SAT(\mathcal{N}')$;
  if $\nexists(i', j')$ **then return** $\mathcal{N}'$;
  if $\mathcal{N}'[i', j'] \neq B$ **then** $repairs \leftarrow repairs \cup \{(i', j')\}$;
  $repairs \leftarrow repairs \cup \{(i', k)\}\ \forall k \in V$ with $i' < k$ s.t. $\mathcal{N}'[i', k] \neq \mathsf{B}$;
  $repairs \leftarrow repairs \cup \{(k, j')\}\ \forall k \in V$ with $k < j'$ s.t. $\mathcal{N}'[k, j'] \neq \mathsf{B}$;
  $repairs \leftarrow repairs \setminus \{(i, j)\}$;

Proposition 5 *Let $\mathcal{N} = (V, C)$ be a QCN. Then, ω is an X-minimal model of $\mathsf{Enc}_1(\mathcal{N})$, with $X = \{q_{ij}^b : i, j \in V$ with $i < j, b \in \mathsf{B}\}$, iff $\mathcal{P} = (V, R)$ is a para-scenario of $\mathcal{N}$ that satisfies $\Delta(\mathcal{P}) \leq \alpha$ and $W(\mathcal{P}) \leq \beta$, where R is defined as follows: $R(i,i) = id$ for every $i \in V$; and $R(i, j) = \bigwedge\{b : \omega(p_{ij}^b) = 1\}$ and $R(j, i) = (R(i, j))^{-1}$ for all $i, j \in V$ with $i < j$.*

For computing one or all X-minimal models, one can use the algorithms proposed in [3].

Now, to solve the optimization problems MinimumDegreePS, MinimumWidthPS, and MinimumBaseRelPS, we need to use Partial MaxSAT.

We use $Enc_2(\mathcal{N}, \beta)$ to denote our encoding for solving MinimumDegreePS. The hard clauses of this encoding come from the formulas $(6) - (8)$ and (11), and the soft clauses are simply the unit clauses $\neg r_{ij}$ for $i, j \in V$ with $i < j$.

Our encoding for solving MinimumBaseRelPS, denoted $Enc_3(\mathcal{N})$, is defined in a similar way to $Enc_2(\mathcal{N}, \beta)$. Its hard clauses come from $(6) - (9)$ and the soft clauses are $\neg q_{ij}^b$ for $b \in \mathsf{B}$ and $i, j \in V$ with $i < j$.

It is worth mentioning that our encoding can easily be adapted to solve the counterparts of the addressed problems that are defined by considering only strong para-scenarios. Indeed, to restrict the search to strong para-scenarios, we only need to restrict our encodings to the base relations occurring in $\mathcal{N}$, i.e., p_{ij}^b and q_{ij}^b exists iff $b \in \mathcal{N}[i, j]$. For instance, the formula (7) is replaced with the following formula:

$$\bigwedge_{i,j,k \in V, i<j<k} \ \bigvee_{(b,b',b'') \in \mathsf{Comp'}} (p_{ik}^b \wedge p_{ij}^{b'} \wedge p_{jk}^{b''}) \qquad (12)$$

where $\mathsf{Comp'} = \mathsf{Comp} \cap (C(i,k) \times C(i,j) \times C(j,k))$

A solution for MinimumWidthPS can be obtained by using $Enc_2(\mathcal{N}, \alpha, \beta)$ in an incremental or a dichotomic search algorithm for finding the minimum value of β.

4.3 Experimentation

We evaluate an in-house implementation of the FINDPARASCENARIO algorithm for solving the FindOnePS problem, called ParaLog, and an implementation of our SAT encoding for solving the MinimumBaseRelPS problem using the PySAT toolkit [21] and the RC2 MaxSAT solver offered there in particular [22]. Regarding ParaLog, we test the following three variants:

- ParaLog^r, which first **r**epairs a QCN via a call to the greedy Algorithm 2 and then extracts an arbitrary consistent atomic QCN from the repaired QCN;
- ParaLog^a, which considers an **a**rbitrary consistent atomic QCN as per the description of the FINDPARASCENARIO algorithm;
- and ParaLog^b, which serves as a **b**aseline and assumes an imaginary consistent atomic QCN with no overlap to the input QCN (so, all of the constraints of the latter will be initially frozen).

We considered RCC8 and IA networks generated by the standard A(n,d,l) model [31], used extensively in literature. In short, A(n,d,l) creates networks of size n, constraint degree d, and an average number l of base relations per constraint. We considered 100 inconsistent networks for *each* average node degree d between 4 and 14 with a 2-degree step and for *each* of the calculi of RCC8 and IA; hence, 1 200 networks in total. For RCC8 we set $n = 30$ and $l = 4.0$, and for IA, $n = 20$ and $l = 6.5$. For this specific range of node degrees d, the networks of model A(n,d,l) lie within the *phase transition* region. The size of the networks is consistent with what has been used in the literature for similar optimization problems in order to present results that are as complete as possible [11, 12] (see Table 2).

The experiments were performed on a computer with an Intel®Xeon®CPU E5-2697 v3 @ 2.60GHz, 264 GB of RAM, and the CentOS Linux 7 OS, using one CPU core per network. All code was implemented and run in Python 3.5; the code is available at: https://msioutis.gitlab.io/software/

The results for the ParaLog variants and the FindOnePS problem, along with the metrics used, are qualitatively similar for both RCC8 and IA, and are shown in Table 1 and Figure 3 for the case of IA (the detailed results for the similar case of RCC8 are omitted to conserve space). The performance of ParaLog^r is superior to the rest of the variants with respect to every important metric used, especially the ones regarding the use of the solver as a SAT oracle and the overall runtime. However, it is interesting to note that all variants are somewhat comparable with regard to the # of extra base relations that they report, with ParaLog^r being only slightly better in general. This indicates that the FINDPARASCENARIO algorithm is good at converging to a good para-scenario, even though it might take it more time to get there due to a bad starting point.

Table 1. FindOnePS: Evaluation with IA networks of model A($n = 20$, d, $l = 6.5$); $\frac{\text{min | avg. | median | max \# of extra base relations}}{\text{avg. \# of visited nodes/avg. \# of oracle calls}}$.

<table>
<tr><th>d</th><th>ParaLog^r</th><th>ParaLog^a</th><th>ParaLog^b</th></tr>
<tr><td>4</td><td>1 | 1.03 | 1 | 2
447.56 | 16.28</td><td>1 | 1 | 1 | 1
1 051.98 | 34.71</td><td>1 | 1.02 | 1 | 2
1 288.9 | 55.98</td></tr>
<tr><td>6</td><td>1 | 1.17 | 1 | 5
334.54 / 18.4</td><td>1 | 1.23 | 1 | 4
1 402.07 / 48.32</td><td>1 | 1.19 | 1 | 4
1 878.38 | 79.62</td></tr>
<tr><td>8</td><td>1 | 2.21 | 1 | 12
362.25 / 37.26</td><td>1 | 2.2 | 1 | 9
1 532.71 / 69.61</td><td>1 | 2.26 | 2 | 10
2 330.78 / 111.39</td></tr>
<tr><td>10</td><td>1 | 5.22 | 4 | 23
663.48 / 87.94</td><td>1 | 5.45 | 4 | 17
1 772.89 / 112.36</td><td>1 | 5.59 | 4 | 19
2 839.26 / 164.35</td></tr>
<tr><td>12</td><td>1 | 10.32 | 9 | 29
1 187.11 / 170.52</td><td>1 | 10.09 | 9 | 26
2 137.75 / 173.46</td><td>1 | 11.73 | 10 | 29
3 628.25 / 248.45</td></tr>
<tr><td>14</td><td>5 | 20.23 | 19 | 39
2 541.93 / 331.37</td><td>8 | 20.83 | 20 | 37
3 377.04 / 298.73</td><td>4 | 23.55 | 23 | 49
5 271.61 / 393.89</td></tr>
</table>

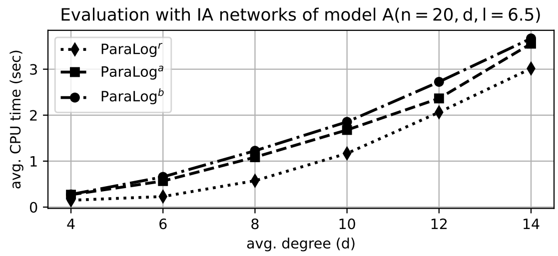

Figure 3. FindOnePS: Runtime for the IA networks of Table 1.

The results for our SAT encoding and the Minimum-BaseRelPS problem are shown in Table 2; again, they are qualitatively similar for both RCC8 and IA, but with starker quantitative differences compared to earlier. The results can be contrasted with the ones obtained by ParaLog as far as paraconsistency levels are concerned, since the SAT encoding provides the optimum ones. Specifically, the denser the QCNs (especially for $d \geq 8$), the more the performance of ParaLog deteriorates with respect to providing low levels of paraconsistency, which is in line with our expectations, and good to have it quantified nonetheless for future research.

Table 2. MinimumBaseRelPS: Evaluation with the concerned RCC8 and IA networks respectively, where a timeout of $1h$ was used for each network; min | avg. | max # of extra base relations • avg. SAT solving time (# of timeouts).

d	RC2(RCC8)	RC2(IA)
4	1 | 1.01 | 2 • $15.44s$ (0)	1 | 1.0 | 1 • $3.47s$ (0)
6	1 | 1.04 | 2 • $22.70s$ (0)	1 | 1.01 | 2 • $11.76s$ (0)
8	1 | 1.22 | 5 • $29.43s$ (0)	1 | 1.22 | 3 • $68.26s$ (1)
10	1 | 1.89 | 6 • $35.67s$ (0)	1 | 1.93 | 4 • $494.14s$ (16)
12	1 | 2.9 | 6 • $83.58s$ (0)	1 | 2.54 | 4 • $1183.81s$ (59)
14	1 | 4.69 | 8 • $476.44s$ (10)	? | ? | ? • inf (100)

5 Inconsistency Measurement

One way to exploit the notion of para-scenario is the definition of inconsistency measures. This is similar to the use of LPm semantics in the definition of contension inconsistency measure in the case of propositional knowledge bases [16].

Inconsistency measures are mappings used to quantify the contradiction in knowledge bases. Most of the works on inconsistency measurement in the literature use postulate-based approaches to capture different aspects relating to inconsistency (e.g. [20, 17, 2, 38]). As far as we know, there are only two studies on inconsistency measurement in qualitative reasoning [13, 33], but they do not offer any inconsistency measure similar to the contension measure. In the realm of temporal reasoning, a recent work has extended this measure to linear temporal logic [14]. Furthermore, independently of qualitative reasoning, [18] introduces measures specifically designed to deal with inconsistent quantitative spatio-temporal information.

We write $\mathbb{R}^+_\infty$ for the set of positive real numbers augmented with a greatest element denoted ∞.

Following the approach for defining inconsistency measures introduced in [20, 9, 13], we consider the following definition.

Definition 6 *An inconsistency measure is a mapping I : QCNs $\rightarrow \mathbb{R}^+_\infty$ that satisfies the following properties for every QCN $\mathcal{N} = (V, C)$:*

- *(Consistency) $I(\mathcal{N}) = 0$ iff $\mathcal{N}$ is consistent;*
- *(Relation Monotonicity) for every QCN $\mathcal{N}'$ with $\mathcal{N}' \subseteq \mathcal{N}$, $I(\mathcal{N}) \leq I(\mathcal{N}')$.*
- *(Variable Monotonicity) for every $V' \subseteq V$, $I(\mathcal{N}_{\downarrow V'}) \leq I(\mathcal{N}')$.*

The inconsistency measures that we introduce are defined as follows:

- $I_1^\beta(\mathcal{N}) = min\{\Delta(\mathcal{P}) : \mathcal{P} \in PSes(\mathcal{N}), W(\mathcal{P}) \leq \beta\}$
- $I_2^\beta(\mathcal{N}) = min\{|\mathcal{P}!| : \mathcal{P} \in PSes(\mathcal{N}), W(\mathcal{P}) \leq \beta\}$

where $PSes(\mathcal{N})$ denotes the set of para-scenarios for $\mathcal{N}$ and $min\ \emptyset = \infty$. One can easily see that similar inconsistency measures can be defined by involving only strong para-scenarios.

In the following proposition, we show that our measures satisfy the desired three properties described in Definition 6.

Proposition 6 *The functions I_1^β and I_2^β are inconsistency measures.*

Proof. We only consider the case of I_1^β, the case I_2^β being similar. Clearly, for every $\mathcal{P} \in PSes(\mathcal{N})$, $0 \leq \Delta(\mathcal{P}) \leq n(n-1)/2$ holds, and it ensues that $0 \leq I_1^\beta(\mathcal{N}) \leq 1$. Moreover, for every $\mathcal{P} \in PSes(\mathcal{N})$, we have $\Delta(\mathcal{N}) = 0$ iff $\mathcal{P}$ is a classical scenario. Thus, we obtain $I_1^\beta(\mathcal{N}) = 0$ iff $\mathcal{N}$ is consistent: I_1^β satisfies Consistency. Let us now show that I_1^β satisfies Relation Monotonicity. Let $\mathcal{N}$ and $\mathcal{N}'$ be two QCNs s.t. $\mathcal{N}' \subseteq \mathcal{N}$ If there is no para-scenario $\mathcal{P}'$ of $\mathcal{N}$ s.t. $W(\mathcal{P}') \leq \beta$, then $I_1^\beta(\mathcal{N}') = 1$, which yields $I(\mathcal{N}) \leq I(\mathcal{N}')$. Otherwise, let $\mathcal{P}'$ be a para-scenario of $\mathcal{N}'$ s.t. $I_1^\beta(\mathcal{N}') = \Delta(\mathcal{P}')/(n(n-1)/2)$. Then $\mathcal{P}$ satisfies Properties 1-4 in Definition 2 for $\mathcal{N}$. Thus, using Minimality, there exists a para-scenario $\mathcal{P}$ of $\mathcal{N}$ s.t. $\Delta(\mathcal{P}) \leq \Delta(\mathcal{P}')$. Consequently, $I(\mathcal{N}) \leq I(\mathcal{N}')$ holds. Variable Monotonicity is mainly a consequence of the fact that for every para-scenario $\mathcal{P}$ of $\mathcal{N}$, the restriction of $\mathcal{P}$ to any subset of variables satisfies Properties 1-4 of para-scenarios. $\dashv$

6 Conclusion and Perspectives

In this paper we have studied a paraconsistency-related approach for inconsistency handling in Qualitative Spatio-Temporal Reasoning. This approach is based on a novel notion called paraconsistent scenario (para-scenario for short), which can be seen as a generalization of the standard notion of scenario by allowing for the existence of multiple base relations between two variables. We have described several interesting theoretical properties of this notion and shown how it can be relevant for inconsistency measurement. We have finally provided and evaluated two distinct open-source approaches for solving the problem of para-scenario computation and other related problems, viz., a constraint freezing-based one and a SAT-based one.

There are several perspectives for future work. Among them, we first mention the study of paraconsistent consequence relations based on para-scenarios, cf. [6, 8]. Additionally, we intend to consider other paraconsistency-related approaches in QSTR. Future work also includes the use of such approaches for measuring inconsistency. Further, there is much room for improvement regarding the scalability of the offered tools via, e.g., designing search heuristics tailored to this problem and providing more compact SAT encodings, cf. [35]. Finally, we would like to explore the use of Answer Set Programming, which has provided very promising results with regard to inconsistency handling in the case of propositional knowledge bases [24, 23].

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